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Abstract

We consider the relativistic transfer equations for photons interacting via emission absorption and scattering with a moving fluid. We prove a minimum-maximum principle and we study the non equilibrium regime: the relativistic correction terms in the scattering operator lead to a frequency drift term modeling the Doppler effects. We prove that the solution of the relativistic transfer equations converges toward the solution of this drift diffusion equation.

1 Introduction

We study some mathematical properties of a system describing the coupling between the relativistic transfer equation for photons and an equation describing the temperature of a fluid moving at the velocity $\vec{u}$. This kind of models was historically derived by physicists ([MWM99, POM05]) and some mathematical properties as existence, uniqueness and a maximum principle have been proved in the non relativistic case [GP86]. The system writes

$$\begin{cases} \dfrac{1}{c}\partial_t I + \vec{\Omega}.\nabla_x I = Q_t & \text{in } [0,T^f]\times\mathbb{R}^3_x\times\mathbb{R}^+_\nu\times S^2, \\ \partial_t T + \nabla.(T\vec{u}) + \Gamma T\nabla.\vec{u} = -c\displaystyle\int_{\nu,\Omega}\frac{\Lambda}{\gamma}Q_t & \text{in } [0,T^f]\times\mathbb{R}^3_x, \end{cases} \tag{1.1}$$

where $I = I(t,x,\nu,\vec{\Omega})$ is the radiative intensity, $T = T(t,x)$ the fluid temperature, $\vec{u} = \vec{u}(x)$ the fluid velocity and $Q_t = Q_t(I,T,\nu)$ describes the interaction between light and fluid, $t \in [0,T^f]$ for a given $0 < T^f < +\infty$ is the time, $x \in \mathbb{R}^3_x$ is the position of the photons, $\nu \in \mathbb{R}^+_\nu$ is the frequency, $\vec{\Omega} \in S^2$ is the direction and c is the speed of light. The relativistic coefficients Λ and γ are

$$\begin{cases} \Lambda = \dfrac{1-\vec{\Omega}.\vec{u}/c}{\sqrt{1-|\vec{u}|^2/c^2}}, \\ \gamma = \dfrac{1}{1-|\vec{u}|^2/c^2}. \end{cases} \tag{1.2}$$

In this work we denote with a subscript 0 the quantities measured in the moving frame, while the others are relative to the ones measured in the reference frame. With these notations, the relation between the frequency ν of a photon measured in the reference frame and its frequency ν_0 measured in the moving frame is

$$\nu_0 = \Lambda\nu. \tag{1.3}$$

In the same way, the relation between the direction $\vec{\Omega}$ of a photon in the reference frame and its direction $\vec{\Omega}_0$ in the moving frame is

$$\vec{\Omega}_0 = \frac{\nu}{\nu_0}\left(\vec{\Omega} - \frac{\gamma}{c}\vec{u}\left(1 - \frac{\vec{\Omega}.\vec{u}}{c}\frac{\gamma}{\gamma+1}\right)\right). \tag{1.4}$$

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A fundamental property is the invariance by change of frame of the photons distribution $f = aI/\nu^3$, where a is a physical constant. This leads to a relation between the radiative intensity I measured in the reference frame and I_0 measured in the moving frame

$$I(\nu, \vec{\Omega}) = \Lambda^{-3} I_0(\nu_0, \vec{\Omega}_0). \tag{1.5}$$

As usual, the operator Q_t consists of the sum of a scattering operator Q_s and an emission absorption operator Q_a. The scattering operator models the diffusion phenomena between light and fluid. We assume that the scattering is coherent (no energy exchange) and isotropic (in the fluid frame). Under these assumptions, Q_s is defined as

$$Q_s = \sigma_s(x)\Lambda\left(\int_{S^2} \frac{\Lambda'}{\Lambda^3} I(\nu', \vec{\Omega}')d\vec{\Omega}' - I\right), \tag{1.6}$$

where, for ease of notations, we defined $\Lambda' = \gamma(1 - \vec{\Omega}'.\vec{u}/c)$ and the measure $d\vec{\Omega}'$ such as $\int_{S^2} d\vec{\Omega}' = 1$. The frequency ν' quantifies the Doppler effects and is defined as $\nu' = \frac{\Lambda}{\Lambda'}\nu$. The coefficient σ_s is the scattering cross section and is assumed to depends only on the position x. We define $Q_{s,0}$ as the scattering operator measured in the moving frame

$$Q_{s,0} = \Lambda^2 Q_s = \sigma_s(x)\left(\int_{S_0^2} I_0 d\vec{\Omega}_0 - I_0\right), \tag{1.7}$$

where, again, the measure $d\vec{\Omega}_0$ is such that $\int_{S_0^2} d\vec{\Omega}_0 = 1$. One recognizes in this expression the classical non relativistic scattering operator. Although the scattering is isotropic in the moving frame, i.e. $\int_{S_0^2} Q_{s,0} d\vec{\Omega}_0 = 0$, this is not true in the reference frame, due to the relativistic effects. The operator Q_a is the emission absorption operator. It is defined as

$$Q_a = \sigma_a(\Lambda\nu)\Lambda\left(\frac{B(\Lambda\nu, T)}{\Lambda^3} - I\right), \tag{1.8}$$

where $B(\nu, T) = \nu^3 \left(e^{\nu/T} - 1\right)^{-1}$ is the (normalized) Planck function. The emission absorption measured in the moving frame is

$$Q_{a,0} = \Lambda^2 Q_a = \sigma_a(\nu_0)\Big(B(\nu_0, T) - I_0\Big). \tag{1.9}$$

The coefficient σ_a is the emission absorption coefficient and is assumed to depend only on the frequency ν. Once again, one recognizes in (1.9) the classical non relativistic emission absorption operator. The derivation of system (1.1) from the coupling between the Euler equations and the relativistic transfer equation is explained for example in [GLG05]: starting from the coupling of the Euler system with the relativistic radiative equation [BD04] and assuming a given fluid density ρ and a given velocity field $\vec{u}$, write the equation of the internal energy by deducting to the equation on the total energy the equation on the kinetic energy. An important property is that this system is not conservative on the physical energy $\int I dx d\nu d\vec{\Omega} + \int T dx$. Actually, one has

$$\frac{d}{dt}\left(\int_{x,\nu,\Omega} I \, dx d\nu d\Omega + \int_x T \, dx\right) + \Gamma \int_x T\nabla.\vec{u} \, dx = \int_{x,\nu,\Omega} (\vec{\Omega}.\vec{u})Q_t \, dx d\nu d\Omega.$$

The first remaining term comes from the hydrodynamic pressure. The second one corresponds to the variation of the kinetic energy of the fluid, which is not taken into account by assuming a given velocity. Due to this non conservation of the energy, the maximum principle proved by F. Golse and B. Perthame in [GP86] for the non relativistic transfer equation does not hold any more, and this leads to mathematical issues.

In this paper we prove two main results. First (section 2), we prove a minimum-maximum principle for the system (1.1) by means of a suitable modification of the Golse and Perthame approach. This result enables to write the system (1.1) as a Lipschitz perturbation of a linear transport equation, from which existence and uniqueness of a solution is obtained. Our point of view is to study the influence of a moving fluid on the radiative intensity, and thus the coefficients σ_a, σ_s and $\vec{u}$ in the system (1.1) will be taken as smooth as necessary. Several theoretical results for more realistic emission absorption coefficient can be found in the non relativistic case in [GP86, BGP87, BGPS88] (see also [YS14] in the context of radiation hydrodynamics). The second main result (section 3) is the proof of convergence with respect to a small parameter ε formally equal to $|\vec{u}|/c$, of the solution of system (1.1) toward the solution of a drift diffusion system in the non-equilibrium regime. This regime is obtained by assuming that the scattering is dominant in comparison with the emission absorption, and that the speed of light is large in comparison with the speed of the fluid. After rescaling with a small parameter ε, it yields the following system

$$\begin{cases} \partial_t I^\varepsilon + \frac{1}{\varepsilon}\vec{\Omega}.\nabla_x I^\varepsilon = \frac{1}{\varepsilon^2} Q_s^\varepsilon + Q_a^\varepsilon & \text{in } [0,T^f]\times\mathbb{R}^3_x\times\mathbb{R}^+_\nu\times S^2, \\ \partial_t T^\varepsilon + \nabla.(T^\varepsilon\vec{u}) + \Gamma T^\varepsilon \nabla.\vec{u} = -\int_{\nu,\Omega} \frac{\Lambda^\varepsilon}{\gamma^\varepsilon}\left(\frac{1}{\varepsilon^2}Q_s^\varepsilon + Q_a^\varepsilon\right) d\nu d\Omega & \text{in } [0,T^f]\times\mathbb{R}^3_x, \end{cases} \tag{1.10}$$

with obvious notations for Q_s^ε and Q_a^ε, and where $\gamma^\varepsilon = \sqrt{1-\varepsilon^2|\vec{u}|^2}^{\,-1}$ and $\Lambda^\varepsilon = \gamma^\varepsilon\big(1-\varepsilon\vec{\Omega}.\vec{u}\big)$. The drift diffusion system writes:

$$\begin{cases} \partial_t\rho - \nabla.\left(\frac{\nabla\rho}{3\sigma_s(x)}\right) + \nabla.(\rho\vec{u}) = \frac{\nabla.\vec{u}}{3}\nu\partial_\nu\rho + \sigma_a(\nu)(B(\nu,T)-\rho), \\ \partial_t T + \nabla.(T\vec{u}) + \Gamma T\nabla.\vec{u} = -\int_\nu \sigma_a(\nu)(B(\nu,T)-\rho)d\nu, \end{cases} \tag{1.11}$$

where $\rho = \lim_{\varepsilon\to 0}\int_{S^2} I^\varepsilon d\Omega$ is the first angular moment of the radiative intensity. This equation has been formally derived by D. Mihalas and B. Weibel Mihalas in [MWM99]. The drift term $\frac{\nabla.\vec{u}}{3}\nu\partial_\nu\rho$ modeling the Doppler effects is also involved in an equation proposed by A. Winslow in [WIN95]. To our knowledge, the mathematical justification of the diffusion system (1.11) that is provided by mean of a convergence result with respect to ε is original. To obtain this convergence result some assumptions will be done on the regularity of the parameters. In particular the emission absorption coefficient will be assumed to belong to $L^2(\mathbb{R}^+_\nu)$, which has no physical meaning, but for technical reasons this assumption is necessary to obtain the convergence in L^2.

In the paper we will use the following notations. The angular integral will be denoted $< . >$, i.e. $< f >= \int_{S^2} f d\Omega$. The space variable x belongs to $\mathbb{R}^3_x$, the frequency variable ν to $\mathbb{R}^+_\nu$, the temperature T to $\mathbb{R}^+_T$ and the time t to $[0,T^f]$, for a given $0 < T^f < +\infty$. We denote by $\|.\|_{L^p_{x,\nu,\Omega}}$ (respectively $\|.\|_{L^p_{t,x}}$) the classical L^p norm on $\mathbb{R}^3_x\times\mathbb{R}^+_\nu\times S^2$ (respectively on $[0,T^f]\times\mathbb{R}^3_x$), for a given $1\le p\le+\infty$. We define the function sgn^+ as

$$\mathrm{sgn}^+(f) = \begin{cases} 1 & f>0, \\ 0 & f\le 0, \end{cases} \tag{1.12}$$

and the positive part f^+ of a function f as $f^+ = f\mathrm{sgn}^+(f)$. The measure in integrals will not be written, i.e. $\int_x . = \int_{\mathbb{R}^3_x} . dx$, $\int_\nu . = \int_{\mathbb{R}_\nu} . d\nu$, $\int_\Omega . = \int_{S^2} . d\Omega$, ...

The paper is organized as follows. In the next section we prove a minimum-maximum principle for the relativistic transfer equation (1.1). The section 3 deals with the non-equilibrium regime and is divided into several parts. In the first one we prove a priori estimates for the drift diffusion system (1.11), such a minimum-maximum principle, some regularity results and we introduce the main result, which is the theorem 3.4 of convergence of (1.10) to (1.11). The next parts deal with the proof of convergence, based on a reconstruction procedure and an original comparison principle and a weight in lemma 3.8. The paper ends with two appendices containing some technical results.

2 A minimum-maximum principle for the relativistic transfer equations

In this section we prove several results for the relativistic transfer equations (1.1). The main result is the following minimum-maximum principle, from which the other results, that is the L^p stability and the fact that it is a Lipschitz perturbation of a linear transport equation, follow easily using classical technics. To prove this minimum-maximum principle we need some minimal assumptions

- (H1) Smoothness of the velocity field: $\vec{u} \in W^{1,\infty}([0,T^f] \times \mathbb{R}^3_x)$. Moreover, $u^* = \|\vec{u}\|_{L^\infty_{t,x}}$ is such that $u^* < c$, where c is the speed of light.
- (H2) Smoothness of the scattering coefficient: $\sigma_s \in W^{1,\infty}(\mathbb{R}^3_x)$ and $\sigma_s > 0$.
- (H3) Smoothness of the emission absorption coefficient: $\sigma_a \in L^\infty(\mathbb{R}^+_\nu)$ and $\sigma_a > 0$.
- (H4) There exists two bounded and positive constants l_* and L_* which are the respectively the infimum and the supremum of the temperature at the initial time: $l_* \leq T(t=0) \leq L_*$. Besides the radiative intensity at the initial time satisfies $B(\nu_0, l_*) \leq I_0(t=0) \leq B(\nu_0, L_*)$.
- (H5) The velocity field is continuous $\vec{u} \in \mathscr{C}^1([0,T^f] \times \mathbb{R}^3_x)$ and the emission absorption coefficient σ_a is integrable, e.i. $\sigma_a \in L^1(\mathbb{R}^+_\nu)$.

The assumption (H5) will be used only to prove that the relativistic transfer system (1.1) is a Lipschitz perturbation of a $\mathscr{C}^0$ semi-group (lemma 2.2). In the forthcoming proofs, we will often use the following bounds, which easily come from the previous assumption on the velocity field, and where the constants $\Lambda_*, \Lambda^* \geq 0$ only depends on u^*:

$$0 < \Lambda_* \leq \Lambda(t,x,\vec{\Omega}) \leq \Lambda^*, \qquad \forall (t,x,\vec{\Omega}) \in [0,T^f] \times \mathbb{R}^3_x \times S^2. \tag{2.1}$$

We introduce our main result:

Theorem 2.1 (Min-max principle)**.** *We assume that hypotheses (H1)-(H4) are satisfied. Then, for all $(t,x,\nu) \in [0,T^f] \times \mathbb{R}^3_x \times \mathbb{R}^+_\nu$, one has the a priori estimates $l(t) \leq T(t) \leq L(t)$ and $B(\nu_0, l(t)) \leq I_0(t) \leq B(\nu_0, L(t))$, where*

$$\begin{cases} l(t) = l_* \exp\left\{ -t\left[(\Gamma+1)\|\vec{u}\|_{W^{1,\infty}_{t,x}} + 2\dfrac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}}{\Lambda_*\sqrt{1-(u^*/c)^2}}\left(1 + \dfrac{2}{c}\dfrac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}}{1-(u^*/c)^2}\right)\right]\right\}, \\ \\ L(t) = L_* \exp\left\{ t\left[(\Gamma+1)\|\vec{u}\|_{W^{1,\infty}_{t,x}} + 2\dfrac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}}{\Lambda_*\sqrt{1-(u^*/c)^2}}\left(1 + \dfrac{2}{c}\dfrac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}}{1-(u^*/c)^2}\right)\right]\right\}. \end{cases} \tag{2.2}$$

This result shows that although the system does not conserve the total energy, due to the absence of kinetic energy balance, energy does not blow up in finite time, and one can give lower and upper bounds on the radiative energy and on the fluid temperature.

Proof. Since the arguments are the same, we only show the proof for the maximum principle. This result is a suitable modification of the maximum principle proved by F. Golse and B. Perthame [GP86] in the non relativistic case. The method is based on varying bounds for which the point is to get the equations that define these bounds. In order to simplify the notations, we denote $B_{0,L} = B(\nu_0, L(t))$. The system (1.1) can be simplified. Actually, using the invariance of the measure $\nu d\nu d\vec{\Omega}$ and the isotropy of the scattering operator in the fluid frame, one has

$$\int_{\nu,\Omega} \frac{\Lambda}{\gamma} Q_t = \int_{\nu,\Omega} \frac{1}{\Lambda\gamma} Q_{t,0} = \int_{\nu_0,\Omega_0} \frac{1}{\gamma} Q_{a,0}, \tag{2.3}$$

and thus system (1.1) reduces to

$$\begin{cases} \dfrac{1}{c}\partial_t I + \vec{\Omega}.\nabla_x I = Q_t & \text{in } [0,T^f]\times\mathbb{R}^3_x\times\mathbb{R}^+_\nu\times S^2, \\ \partial_t T + \nabla.(T\vec{u}) + \Gamma T\nabla.\vec{u} = -\dfrac{c}{\gamma}\displaystyle\int_{\nu_0,\Omega_0} Q_{a,0} & \text{in } [0,T^f]\times\mathbb{R}^3_x. \end{cases} \tag{2.4}$$

Multiplying the first equation of (1.1) by $c\frac{\Lambda}{\gamma}\text{sgn}^+(I_0 - B_{0,L})$, integrating over $\mathbb{R}^+_\nu \times S^2$ and reminding that $I(\nu,\vec{\Omega}) = \Lambda^{-3} I_0(\nu_0,\vec{\Omega}_0)$, one can deduce

$$\begin{aligned}&\int_{\nu,\Omega} \frac{\text{sgn}^+(I_0 - B_{0,L})}{\Lambda^2\gamma}\Big(\partial_t I_0 + c\vec{\Omega}.\nabla I_0\Big) + \int_{\nu,\Omega}\text{sgn}^+(I_0 - B_{0,L}) I_0 \frac{\Lambda}{\gamma}\Big(\partial_t\Lambda^{-3} + c\vec{\Omega}.\nabla\Lambda^{-3}\Big)\\ =&c\int_{\nu,\Omega}\frac{\Lambda\text{sgn}^+(I_0 - B_{0,L})}{\gamma}Q_t.\end{aligned}$$

Developing the derivatives of Λ^{-3} and using the invariance of the measure $\nu d\nu d\vec{\Omega}$, one can get after algebraic manipulations

$$\begin{aligned}&\int_{\nu,\Omega}\Big(\partial_t\frac{(I_0 - B_{0,L})^+}{\Lambda^2\gamma} + c\vec{\Omega}.\nabla\frac{(I_0 - B_{0,L})^+}{\Lambda^2\gamma}\Big) + \int_{\nu,\Omega}\frac{\text{sgn}^+(I_0 - B_{0,L})}{\Lambda^2\gamma}\Big(\partial_t B_{0,L} + c\vec{\Omega}.\nabla B_{0,L}\Big)\\ &\qquad - \int_{\nu,\Omega}(I_0 - B_{0,L})^+\Big(\partial_t\frac{1}{\Lambda^2\gamma} + c\vec{\Omega}.\nabla\frac{1}{\Lambda^2\gamma}\Big) - 3\int_{\nu,\Omega}\frac{(I_0 - B_{0,L})^+}{\Lambda^3\gamma}\Big(\partial_t\Lambda + c\vec{\Omega}.\nabla\Lambda\Big)\\ &\qquad - 3\int_{\nu,\Omega}B_{0,L}\frac{\text{sgn}^+(I_0 - B_{0,L})}{\Lambda^3\gamma}\Big(\partial_t\Lambda + c\vec{\Omega}.\nabla\Lambda\Big) = \frac{c}{\gamma}\int_{\nu_0,\Omega_0} sgn^+(I_0 - B_{0,L})Q_{t,0}.\end{aligned}$$

Regrouping the terms of the second line together, it yields after rearrangements

$$\begin{aligned}&\int_{\nu,\Omega}\Big(\partial_t\frac{(I_0 - B_{0,L})^+}{\Lambda^2\gamma} + c\vec{\Omega}.\nabla\frac{(I_0 - B_{0,L})^+}{\Lambda^2\gamma}\Big) - \int_{\nu,\Omega}(I_0 - B_{0,L})^+\Lambda^{-3}\Big(\partial_t\frac{\Lambda}{\gamma} + c\vec{\Omega}.\nabla\frac{\Lambda}{\gamma}\Big)\\ =& \int_{\nu,\Omega}\frac{\text{sgn}^+(I_0 - B_{0,L})}{\Lambda^2\gamma}\Big\{3B_{0,L}\Big(\frac{\partial_t\Lambda}{\Lambda} + c\frac{\vec{\Omega}.\nabla\Lambda}{\Lambda}\Big) - \partial_t B_{0,L} - c\vec{\Omega}.\nabla B_{0,L}\Big\}\\ &+ \frac{c}{\gamma}\int_{\nu_0,\Omega_0}\text{sgn}^+(I_0 - B_{0,L})Q_{t,0}.\end{aligned}$$

We remind that $B(\nu_0, L(t)) = \frac{\nu_0^3}{e^{\nu_0/L(t)}-1} = \frac{(\Lambda\nu)^3}{e^{\Lambda\nu/L(t)}-1}$. It yields

$$\partial_t B(\nu_0, L(t)) = \frac{3\Lambda^2\nu^3\partial_t\Lambda}{e^{\nu_0/L(t)}-1} - \frac{\nu L\partial_t\Lambda - \nu_0\partial_t L}{L^2}\frac{e^{\nu_0/L(t)}}{e^{\nu_0/L(t)}-1}\frac{\nu_0^3}{e^{\nu_0/L(t)}-1},$$

which can be written $\partial_t B(\nu_0, L(t)) = B(\nu_0, L(t))\big(3\partial_t(\log\Lambda) - \frac{\nu_0}{L(t)}\frac{\partial_t(\log\Lambda) - \partial_t(\log L(t))}{1-e^{-\nu_0/L(t)}}\big)$. The same manipulations lead to $\nabla B(\nu_0, L(t)) = B(\nu_0, L(t))\big(3\nabla(\log\Lambda) - \frac{\nu_0}{L(t)}\frac{\nabla(\log\Lambda)}{1-e^{-\nu_0/L(t)}}\big)$. The previous equation thus becomes

$$\begin{aligned}&\int_{\nu,\Omega}\partial_t\frac{(I_0 - B_{0,L})^+}{\Lambda^2\gamma} + \int_{\nu,\Omega}c\vec{\Omega}.\nabla\frac{(I_0 - B_{0,L})^+}{\Lambda^2\gamma} = \frac{c}{\gamma}\int_{\nu_0,\Omega_0}\text{sgn}^+(I_0 - B_{0,L})Q_{t,0}\\ &\qquad + \int_{\nu,\Omega}\frac{\text{sgn}^+(I_0 - B_{0,L})}{L(t)\Lambda^2\gamma}B_{0,L}\frac{\nu_0}{1-e^{-\nu_0/L(t)}}\Big(\partial_t(\log\Lambda) + c\vec{\Omega}.\nabla(\log\Lambda) - \partial_t(\log L(t))\Big)\\ &\qquad + \int_{\nu,\Omega}(I_0 - B_{0,L})^+\Lambda^{-3}\Big(\partial_t\frac{\Lambda}{\gamma} + c\vec{\Omega}.\nabla\frac{\Lambda}{\gamma}\Big).\end{aligned} \tag{2.5}$$

We now turn to the study of the second equation of (1.1). We make the same manipulations than for the equation for I: multiplying it by $\mathrm{sgn}^+(T-L(t))$ and integrating over $\mathbb{R}^+_x$ yields

$$\frac{d}{dt}\int_x (T-L(t))^+ + \int_x \mathrm{sgn}^+(T-L(t))\big(\nabla.(T\vec{u}) + \Gamma T\nabla.\vec{u}\big) = -\frac{c}{\gamma}\int_{x,\nu_0,\Omega_0} \mathrm{sgn}^+(T-L(t))Q_{a,0}.$$

One has $\nabla.(T\vec{u})+\Gamma T\nabla.\vec{u} = (T-L(t))\nabla.\vec{u}+\vec{u}.\nabla(T-L(t))+\Gamma(T-L(t))\nabla.\vec{u}+L(t)\nabla.\vec{u}+\Gamma L(t)\nabla.\vec{u}$. Thus, $\mathrm{sgn}^+(T-L(t))\big(\nabla.(T\vec{u})+\Gamma T\nabla.\vec{u}\big) = \nabla.\big((T-L(t))^+ + \Gamma(T-L(t))^+\nabla.\vec{u} + \mathrm{sgn}^+(T-L(t))(\Gamma+1)L(t)\nabla.\vec{u}$. We thus have

$$\begin{aligned}\frac{d}{dt}\int_x (T-L(t))^+ + \Gamma\int_x (T-L(t))^+\nabla.\vec{u} + \int_x \mathrm{sgn}^+(T-L(t))\Big(\partial_t L(t) + (\Gamma+1)\nabla.\vec{u}L(t)\Big)& \\ = -\frac{c}{\gamma}\int_{x,\nu_0,\Omega_0}\mathrm{sgn}^+(T-L(t))Q_{a,0}.&\end{aligned} \tag{2.6}$$

Let us study the source terms involved in (2.5). By definition of the scattering operator (1.7), one has

$$\int_{\nu_0,\Omega_0}\mathrm{sgn}^+(I_0-B_{0,L})Q_{s,0} = \sigma_s(x)\int_{\nu_0,\Omega_0}\mathrm{sgn}^+(I_0-B_{0,L})\bigg(\int_{\Omega_0'} I_0(\vec{\Omega}_0') - I_0\bigg),$$

which we write

$$\int_{\nu_0,\Omega_0}\mathrm{sgn}^+(I_0-B_{0,L})Q_{s,0} = -\sigma_s(x)\int_{\nu_0}\bigg(\big\langle (I_0-B_{0,L})^+\big\rangle_0 - \big\langle I_0-B_{0,L}\big\rangle_0\big\langle sgn^+(I_0-B_{0,L})\big\rangle_0\bigg),$$

where the notation $\langle . \rangle_0$ means $\int_{S_0^2} . \, d\vec{\Omega}_0$. Given a function $X\in L^1$ and denoting X^- its non positive part, one has $< X > = < X^+ + X^- > \leq < X^+ > = < X\mathrm{sgn}^+(X) >$. Multiplying this identity by $< \mathrm{sgn}^+(X) >$ and using $< \mathrm{sgn}^+(X) > \leq 1$, one obtains $< X >< \mathrm{sgn}^+(X) > \leq < X^+ >$, which yields the fact that $\int_{\nu_0,\vec{\Omega}_0}\mathrm{sgn}^+(I_0-B_{0,L})Q_{s,0}\leq 0$. We now turn to the terms containing the emission absorption coefficients. By definition of the emission absorption operator (1.9), one has

$$\begin{aligned}&\int_{\nu_0,\Omega_0}\big(\mathrm{sgn}^+(I_0-B_{0,L}) - \mathrm{sgn}^+(T-L(t))\big)Q_{a,0} \\ &\qquad = \int_{\nu_0,\Omega_0}\sigma_a(\nu_0)\big(B_{0,T}-I_0\big)\Big(\mathrm{sgn}^+(I_0-B_{0,L}) - \mathrm{sgn}^+(T-L(t))\Big).\end{aligned} \tag{2.7}$$

The proof of the fact that this term is non positive is done in [GP86, DO01]. Actually, equation (2.7) can be written

$$\begin{aligned}&\int_{\nu,\Omega}\big(\mathrm{sgn}^+(I_0-B_{0,L}) - \mathrm{sgn}^+(T-L(t))\big)Q_{a,0} \\ &\qquad = -\int_{\nu_0,\Omega_0}\sigma_a(\nu_0)\Big((I_0-B_{0,L})^+ - (I_0-B_{0,L})\Big)\mathrm{sgn}^+(T-L(t)) \\ &\qquad\quad -\int_{\nu_0,\Omega_0}\sigma_a(\nu_0)\Big((T-L(t))^+ - (T-L(t))\Big)\mathrm{sgn}^+(I_0-B_{0,L}),\end{aligned}$$

and thus this term is negative since the function $T\mapsto B(\nu,T)$ is non decreasing. Integrating equations (2.6) on $\mathbb{R}^3$ and adding equation (2.6), one gets, using all these results

$$\begin{aligned}&\frac{d}{dt}\int_x\bigg(\int_{\nu,\Omega}\frac{(I_0-B_{0,L})^+}{\Lambda^2\gamma} + (T-L(t))^+\bigg) \leq -\mathrm{sgn}^+(T-L(t))\Big(\partial_t L(t) + (\Gamma+1)\nabla.\vec{u}L(t)\Big) \\ &+\int_{\nu,\Omega}\frac{\mathrm{sgn}^+(I_0-B_{0,L})}{L(t)\Lambda^2\gamma}B_{0,L}\frac{\nu_0}{1-e^{-\nu_0/L(t)}}\Big(\partial_t(\log\Lambda) + c\vec{\Omega}.\nabla(\log\Lambda) - \partial_t(\log L(t))\Big) \\ &+\ \max\left(\left\|\frac{\partial_t\Lambda\gamma^{-1} + c\vec{\Omega}.\nabla\Lambda\gamma^{-1}}{\Lambda^3}\right\|_{L^\infty_{x,\vec{\Omega}}},\ \Gamma\|\nabla.\vec{u}\|_{L^\infty_x}\right)\int_x\bigg(\int_{\nu,\Omega}(I_0-B_{0,L})^+ + (T-L(t))^+\bigg).\end{aligned} \tag{2.8}$$

The key of the proof is that by definition of L(t), the right member of the first line and the second line of this inequality are non positive. Let us check this. By definition of $L(t)$ (2.2), one has

$$\frac{\partial_t L(t)}{L(t)} = \big(\Gamma+1\big)\|\vec{u}\|_{W^{1,\infty}_{t,x}} + 2\frac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}}{\Lambda_*\sqrt{1-(u^*/c)^2}}\left(1+\frac{2}{c}\frac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}}{1-(u^*/c)^2}\right).$$

In particular,

$$\begin{cases} \partial_t L(t)/L(t) \geq (\Gamma+1)\|\vec{u}\|_{W^{1,\infty}_{t,x}}, \\ \partial_t L(t)/L(t) \geq \dfrac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}}{\Lambda_*\sqrt{1-(u^*/c)^2}}\left(1+\dfrac{2}{c}\dfrac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}}{1-(u^*/c)^2}\right). \end{cases} \tag{2.9}$$

The first inequality in (2.9) yields the non positivity of the right member of the first line of (2.8). For the second line of (2.8), the definition of Λ (1.2) yields

$$\partial_t\Lambda = \frac{\vec{\Omega}.\partial_t\vec{u}/c}{\sqrt{1-|\vec{u}|^2/c^2}} + \left(1-\vec{\Omega}.\frac{\vec{u}}{c}\right)\frac{\vec{u}.\partial_t\vec{u}/c^2}{\sqrt{1-|\vec{u}|^2/c^2}^3}.$$

Simple computations using the assumption (H1) on the velocity field and the estimate (2.1) lead to

$$\Lambda^{-1}\partial_t\Lambda \leq \frac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}/c}{\Lambda_*\sqrt{1-(u^*/c)^2}}\left(1+2\frac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}/c}{1-(u^*/c)^2}\right).$$

The same manipulations for the space derivatives of Λ give us

$$\partial_t(\log\Lambda) + c\vec{\Omega}.\nabla(\log\Lambda) \leq 2\frac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}}{\Lambda_*\sqrt{1-(u^*/c)^2}}\left(1+\frac{2}{c}\frac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}}{1-(u^*/c)^2}\right).$$

The second inequality of (2.9) thus gives the non positivity of the second line of (2.8). Equation (2.8) thus reduces to

$$\frac{d}{dt}\int_x\left(\int_{\nu,\Omega}\frac{\big(I_0-B_{0,L}\big)^+}{\Lambda^2\gamma} + \big(T-L\big)^+\right) \leq C\int_x\left(\int_{\nu,\Omega}\big(I_0-B_{0,L}\big)^+ + \big(T-L\big)^+\right),$$

where $C = \max\left(\left\|\frac{\partial_t\Lambda\gamma^{-1}+c\vec{\Omega}.\nabla\Lambda\gamma^{-1}}{\Lambda^3}\right\|_{L^\infty_{x,\vec{\Omega}}}, \Gamma\|\nabla.\vec{u}\|_{L^\infty_x}\right)$. Integrating this inequality between 0 and t and using the positivity of the coefficients Λ and γ (estimate (2.1) and assumption (H1)) yields

$$\begin{aligned} &\min\left((\Lambda^*)^{-2}\inf_{t,x}\gamma^{-1},1\right)\int_x\left(\int_{\nu,\Omega}(I_0-B_{0,L})^+(t) + (T-L)^+(t)\right) \\ &\leq \int_x\left(\int_{\nu,\Omega}\frac{(I_0-B_{0,L})^+(0)}{\Lambda^2\gamma} + (T-L)^+(0)\right) + \int_0^t C\int_x\left(\int_{\nu,\Omega}(I_0-B_{0,L})^+ + (T-L)^+\right). \end{aligned}$$

The Gronwall lemma thus gives

$$\int_x\left(\int_{\nu,\Omega}\big(I_0-B_{0,L}\big)^+(t) + \big(T-L\big)^+(t)\right) \leq C_2\int_x\left(\int_{\nu,\Omega}\frac{(I_0-B_{0,L})^+(0)}{\Lambda^2\gamma} + (T-L)^+(0)\right)e^{t\frac{\|C\|_{L^\infty_t}}{C_2}},$$

where $C_2 = \min\left((\Lambda^*)^{-2}\inf_{t,x}\gamma^{-1},1\right)^{-1}$. The assumption (H4) on the initial condition thus yields that the left member is non positive, which is the result of the claim. □

The proof of the minimum principle is similar. Writing the evolution equation satisfied by $\int_x\left(\int_{\nu,\Omega}\frac{(B_{0,l}-I_0)^+}{\Lambda^2\gamma} + (l-T)^+\right)$, the claim relies on the following inequalities satisfied by $l(t)$

$$\begin{cases} \partial_t l(t)/l(t) \leq -(\Gamma+1)\|\vec{u}\|_{W^{1,\infty}_{t,x}}, \\ \partial_t l(t)/l(t) \leq -\dfrac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}}{\Lambda_*\sqrt{1-(u^*/c)^2}}\left(1+\dfrac{2}{c}\dfrac{\|\vec{u}\|_{W^{1,\infty}_{t,x}}}{1-(u^*/c)^2}\right). \end{cases}$$

To obtain existence of solutions of (1.1), one can use the semi-group theory for problem written as Lipschitz perturbation of semi-group operators. In the following lemma the notion of strong and classical solution is the one used by A. Pazy [PAZ83].

Lemma 2.2. *Assume that hypotheses (H1)-(H5) are satisfied. Then system* (1.1) *has a unique solution* $(I,T) \in \mathscr{C}^0([0,T^f];L^p(\mathbb{R}^3_x \times \mathbb{R}^+_\nu \times S^2)) \times \mathscr{C}^0([0,T^f];L^p(\mathbb{R}^3_x))$*. The solution is strong in the case* $1 < p < +\infty$ *and classical in the case* $p = 1$*.*

Proof. Let us rewrite the system (1.1) as an evolution system involving several operators, that is

$$\begin{cases} \partial_t(T,I) + Q(T,I) = Q_{lips}(T), \\ (T,I)(t=0) = (T^{in}, I^{in}) \end{cases} \tag{2.10}$$

where $Q(T,I) = A(T,I) - Q_{lin}(T,I)$,

$$\begin{cases} A(T,I) = \Big(\vec{u}.\nabla T \; ; \; c\vec{\Omega}.\nabla I\Big), \\ Q_{lin}(T,I) = \Big(-(1+\Gamma)T\nabla.\vec{u} + \dfrac{c}{\gamma}\displaystyle\int_{\nu,\Omega} \dfrac{\sigma_a(\nu_0)}{\Lambda} I_0 \; ; \; cQ_s - c\dfrac{\sigma_a(\nu_0)}{\Lambda^2} I_0\Big), \\ Q_{lips}(T) = c\Big(-\displaystyle\int_{\nu,\Omega} \dfrac{\sigma_a(\nu_0)}{\gamma\Lambda^2} B(\nu_0,T) \; ; \; \dfrac{\sigma_a(\nu_0)}{\Lambda^2} B(\nu_0,T)\Big), \end{cases}$$

where Q is the generator of the semi-group and Q_{lips} is the perturbation. The result of the claim is a consequence of the two following lemmas. In lemma 2.3 we prove that Q is the infinitesimal generator of a $\mathscr{C}^0$ semigroup on $L^p(\mathbb{R}^3_x) \times L^p(\mathbb{R}^3_x \times \mathbb{R}^+_\nu \times S^2)$ and in lemma 2.4 we prove that Q_{lips} is a Lipschitz operator from $L^p(\mathbb{R}^3_x) \times L^p(\mathbb{R}^3_x \times \mathbb{R}^+_\nu \times S^2)$ into itself. One then applies theorems 6.1.2 and 6.1.6 of [PAZ83] in the case $1 < p < +\infty$ and theorems 6.1.2 and 6.1.5 of [PAZ83] in the case $p = 1$. The difference comes from the fact that L^p is a reflexive Banach space only in the case $1 < p < +\infty$. □

Lemma 2.3. *Assume that hypotheses (H1)-(H5) are satisfied. Then, for all* $1 \le p < +\infty$*, the operator* $Q = A - Q_{lin}$ *is the infinitesimal generator of a* $\mathscr{C}^0$ *semigroup on* $L^p(\mathbb{R}^3_x) \times L^p(\mathbb{R}^3_x \times \mathbb{R}^+_\nu \times S^2)$*.*

Proof. It is known (see [DL83]) that A is the infinitesimal generator of a $\mathscr{C}^0$ semigroup on $L^p(\mathbb{R}^3_x) \times L^p(\mathbb{R}^3_x \times \mathbb{R}^+_\nu \times S^2)$. We need to prove that Q_{lin} is a linear continuous operator from $L^p(\mathbb{R}^3_x) \times L^p(\mathbb{R}^3_x \times \mathbb{R}^+_\nu \times S^2)$ into itself. Using the inequality $\forall a,b \ge 0, (a+b)^p \le 2^{p-1}(a^p + b^p)$, we just need to estimate each components of Q_{lin} in L^p. We start with the first component of Q_{lin}. One has $\|(1+\Gamma)T\nabla.\vec{u}\|_{L^p_x} \le (1+\Gamma)\|\nabla.\vec{u}\|_{L^\infty_x}\|T\|_{L^p_x}$. For the second term of the first component, the relation $I_0 = \Lambda^3 I$ and the estimate (2.1) give

$$\left\|\frac{c}{\gamma}\int_{\nu,\Omega} \frac{\sigma_a(\nu_0)}{\Lambda} I_0\right\|^p_{L^p_x} \le c^p(\Lambda^*)^{2p}\|\gamma^{-p}\|_{L^\infty_x} \int_x \left(\int_{\nu,\Omega} \sigma_a(\nu_0) I\right)^p.$$

By assumption (H1) on the velocity field, one has $\gamma^{-1} \le 1$. Using the Hölder inequality, we get

$$\left\|\frac{c}{\gamma}\int_{\nu,\Omega} \frac{\sigma_a(\nu_0)}{\Lambda} I_0\right\|_{L^p_x} \le c\Lambda^{*2}\|\sigma_a\|_{L_\nu^{\frac{p}{p-1}}}\|I\|_{L^p_{x,\nu,\Omega}}, \tag{2.11}$$

which is bounded thanks to assumption $(H5)$ on the regularity of the emission absorption coefficient. The second component is a little more complicated. We remind that

$$Q_s = \sigma_s(x)\Lambda\Big(\int_{\Omega'} \frac{\Lambda'}{\Lambda^3} I(\nu', \vec{\Omega}') - I\Big).$$

Thus, a Cauchy Schwarz inequality and the estimate (2.1) yield

$$\|Q_s\|^p_{L^p_{x,\nu,\Omega}} \leq \|\sigma_s\|^p_{L^\infty_x}(\Lambda^*)^p\Bigg(\left(\frac{\Lambda^*}{\Lambda^3_*}\right)^p\int_{x,\nu,\Omega} I^p(\nu',\vec{\Omega}') + \|I\|^p_{L^p_{x,\nu,\Omega}}\Bigg).$$

Making the change of variable $\bar{\nu} = \frac{\Lambda}{\Lambda'}\nu$ in the integral, we find

$$\|Q_s\|^p_{L^p_{x,\nu,\Omega}} \leq \|\sigma_s\|^p_{L^\infty_x}(\Lambda^*)^p\Bigg(\left(\frac{\Lambda^*}{\Lambda^3_*}\right)^{p+1} + 1\Bigg)\|I\|^p_{L^p_{x,\nu,\Omega}},$$

which yields $\|Q_s\|_{L^p_{x,\nu,\Omega}} \leq C\|I\|_{L^p_{x,\nu,\Omega}}$. The second term of the second component is similar to (2.11). We deduce that if we denote $F = (T, I)$, there exists a constant $C \geq 0$ such that $\|Q_{lin}(F)\|_{L^p_x\times L^p_{x,\nu,\Omega}} \leq C\|F\|_{L^p_x\times L^p_{x,\nu,\Omega}}$. There just remains to apply theorem 3.1.1 of [PAZ83] to conclude. □

We study the operator Q_{lips}. We have the

Lemma 2.4. *Under hypotheses (H1)-(H4), there exists a constant $C \geq 0$ such that for all T_1, $T_2 \in \left(L^p(\mathbb{R}^3_x) \cap L^\infty(\mathbb{R}^3_x)\right)^+$, with $T_1 \leq T_2$, the following estimate holds :*

$$\|Q_{lips}(T_1) - Q_{lips}(T_2)\|_{L^p_x\times L^p_{x,\nu,\Omega}} \leq C\|T_1 - T_2\|_{L^p_x}.$$

Proof. Studying the expression of $Q_{lips}(T_1)-Q_{lips}(T_2)$, we see that we need to estimate $\|B(\nu_0, T_1)-B(\nu_0, T_2)\|_{L^p_{x,\nu,\Omega}}$. Making a Taylor expansion of the function $T \mapsto B(\nu_0, T)$, one gets

$$|B(\nu_0, T_2) - B(\nu_0, T_1)|^p = \left|\int_{T_1}^{T_2}\partial_T B(\nu_0, s)ds\right|^p \leq |T_1 - T_2|^p\left|\frac{1}{|T_2 - T_1|}\int_{T_1}^{T_2}\partial_T B(\nu_0, s)ds\right|^p.$$

The Jensen inequality yields

$$|B(\nu_0, T_2) - B(\nu_0, T_1)|^p \leq |T_1 - T_2|^p\frac{1}{|T_2 - T_1|}\int_{T_1}^{T_2}|\partial_T B(\nu_0, s)|^p ds. \tag{2.12}$$

By definition of the Planck function, one has $\partial_T B(\nu_0, s) = \frac{\nu_0^4 e^{\nu_0/s}}{s^2(e^{\nu_0/s}-1)^2}$. Integrating (2.12) on $\mathbb{R}^+_\nu$ and making the change of variable $\nu \to \nu_0/s$ leads to

$$\int_\nu |B(\nu_0, T_2) - B(\nu_0, T_1)|^p \leq |T_1 - T_2|^p\frac{1}{\Lambda}\frac{1}{|T_2 - T_1|}\int_{T_1}^{T_2} s^{2p+1}\int_\nu\frac{\nu^{4p}e^{p\nu}}{(e^\nu - 1)^{2p}}dsd\nu.$$

Since there exists a constant C such that $\int_\nu \frac{\nu^{4p}e^{p\nu}}{(e^\nu-1)^{2p}}dsd\nu \leq C$, this expression reduces to

$$\int_\nu |B(\nu_0, T_2) - B(\nu_0, T_1)|^p \leq |T_1 - T_2|^p\frac{C}{\Lambda(2p+2)}\frac{T_2^{2p+2} - T_1^{2p+2}}{|T_2 - T_1|}.$$

Using the formula $a^n - b^n = (a-b)\sum_{k=0}^{n-1}a^k b^{n-1-k}$, one finds a constant C such that

$$\|B(\nu_0, T_2) - B(\nu_0, T_1)\|_{L^p_{x,\nu,\Omega}} \leq C\|T_2 - T_1\|_{L^p_x}.$$

This is the same idea to prove the result for the first component of Q_{lips}. It is an easy matter to conclude. □

3 Non-equilibrium regime

In this section we study the so called non-equilibrium regime. This regime has already been studied in the grey case with relativistic coefficients in [GLG05] and with non relativistic coefficients in [DO01] (see also [BN14]). The idea is to assume that the speed of light c is very fast compared to the velocity field $\vec{u}$, i.e. $|\vec{u}|/c << 1$ and that the scattering coefficient is stiff compared to the emission absorption coefficient, i.e. $\sigma_s/\sigma_a >> 1$. We thus introduce a coefficient ε, $0 < \varepsilon \leq 1$, formally equal to the ratio of a characteristic speed of the fluid by the velocity of light (a rigorous derivation of the equations can be found for example in [GLG05, BD04]). Rescaling the emission absorption coefficient as $\sigma_a = \varepsilon^{-1}\widehat{\sigma_a}$ and the scattering coefficient as $\sigma_s = \varepsilon\widehat{\sigma_s}$ leads, after dropping the hats for ease of notations, to the following system

$$\begin{cases} \partial_t I^\varepsilon + \dfrac{1}{\varepsilon}\vec{\Omega}.\nabla_x I^\varepsilon = \dfrac{Q_s^\varepsilon}{\varepsilon^2} + Q_a^\varepsilon, \\ \partial_t T^\varepsilon + \nabla.(T^\varepsilon \vec{u}) + \Gamma T^\varepsilon \nabla.\vec{u} = -\displaystyle\int_{\nu,\Omega} \dfrac{\Lambda^\varepsilon}{\gamma^\varepsilon} Q_a^\varepsilon, \end{cases} \tag{3.1}$$

where $\gamma^\varepsilon = \left(1 - \varepsilon^2|\vec{u}|^2\right)^{-1/2}$ and $\Lambda^\varepsilon = \gamma^\varepsilon\left(1 - \varepsilon\vec{\Omega}.\vec{u}\right)$. We introduce $(\rho, \bar{T})$ the solution of the following drift diffusion system:

$$\begin{cases} \partial_t \rho - \nabla.\left(\dfrac{\nabla\rho}{3\sigma_s(x)}\right) + \nabla.\left(\rho\vec{u}\right) = \dfrac{\nabla.\vec{u}}{3}\nu\partial_\nu\rho + \sigma_a(\nu)(B(\nu,\bar{T}) - \rho), \\ \partial_t \bar{T} + \nabla.(\bar{T}\vec{u}) + \Gamma\bar{T}\nabla.\vec{u} = -\displaystyle\int_\nu \sigma_a(\nu)(B(\nu,\bar{T}) - \rho). \end{cases} \tag{3.2}$$

This section is devoted to the proof of convergence of the solution $(I^\varepsilon, T^\varepsilon)$ of the relativistic transfer equations (3.1) to the solution of the drift diffusion system (3.2) as $\varepsilon \to 0$. The proof will be done in three steps, following C. Dogbe [DO01] or G. Allaire and F. Golse [AG12]. The idea is in a first time to find formally the limit, as $\varepsilon \to 0$, of the solution $(I^\varepsilon, T^\varepsilon)$ using formal Hilbert expansions. In a second time a function is reconstructed from the truncated Hilbert expansion, solution of the system (3.1) with a remainder. Finally, we conclude by using a priori estimates on the solution of the drift diffusion system (3.2) and a stability result for the system (3.1).

This section is organized as follows. In a first part the drift diffusion system will be derived using formal Hilbert expansion of the radiative transfer equations. In a second part a rigorous convergence result is proved. Some technical results are postponed to the appendix.

3.1 Formal asymptotic of the radiative transfer equations

In this part the drift diffusion system (3.2) will be obtained formally, using formal Hilbert expansions of the radiative transfer equations (3.1). Indeed, we prove the following lemma.

Lemma 3.1. *The formal limit of the solution of the transfer equations* (3.1) *as* ε *tends to 0 is solution of the drift diffusion system* (3.2).

Proof. The proof is divided in two steps. In a first one the scattering and emission absorption operators will be expended in power of ε. In a second part the solution of the radiative transfer equations (3.1) will also be expended, leading formally to the drift diffusion system (3.2).

3.1.1 First step: Study of the source terms

In order to simplify the next step, concerning the Hilbert expansion of the solution $(I^\varepsilon, T^\varepsilon)$ of the system (3.1), the scattering and the emission absorption operators are expended in power of ε. Since it is of order ε^{-2}, the study of the scattering operator will be more complicated, while the expansion of the emission absorption operator will be rather simple. We start with the scattering operator. Given a function $I : [0, T^f] \times \mathbb{R}^3_x \times \mathbb{R}^+_\nu \times S^2 \to \mathbb{R}^+$, it is defined by

$$Q_s^\varepsilon(I) = \sigma_s(x)\Lambda^\varepsilon\Big(\int_{S^2} \frac{\Lambda^{\varepsilon'}}{(\Lambda^\varepsilon)^3} I(\nu^{\varepsilon'}, \vec{\Omega}')d\vec{\Omega}' - I\Big),$$

where $\nu^{\varepsilon'} = (\Lambda^\varepsilon/\Lambda^{\varepsilon'})\nu$. We expend it in power of ε: given $N \in \mathbb{N}$, we write

$$Q_s^\varepsilon(I) = \sum_{0\le i\le N} \varepsilon^i Q_s^i(I) + \varepsilon^{N+1}\bar{Q}^\varepsilon_{s,N}(I).$$

Due to the Doppler shift, the radiative intensity I is computed at the frequency ν', and a Taylor expansion with integral remainder will be performed. In order to simplify the notations we remove the dependence in ε in the coefficients $\Lambda^\varepsilon, \Lambda^{\varepsilon'}$ and $\nu^{\varepsilon'}$. Since il will be useful in the next part, expansions at order 2,1 and 0 with respect to ε of the scattering operator are performed.

Expansion of Q_s^ε at order 2

In this part the scattering operator is written as

$$Q_s^\varepsilon(I) = Q_s^0(I) + \varepsilon Q_s^1(I) + \varepsilon^2 Q_s^2(I) + \varepsilon^3 \bar{Q}^\varepsilon_{s,2}(I),$$

with

$$\begin{cases} Q_s^0(I) = \sigma_s(x)(< I > -I) \\[2ex] Q_s^1(I) = \sigma_s(x)\Big(\displaystyle\int_{S^2} \Big(\nu\lambda_5\partial_\nu I(\nu,\vec{\Omega}') + \lambda_3 I(\nu,\vec{\Omega}')\Big)d\vec{\Omega}' + \lambda_1\sigma_s(x)(< I > -I)\Big) \\[2ex] Q_s^2(I) = \sigma_s(x)\displaystyle\int_{S^2}\Big(\nu(\lambda_6+\lambda_3\lambda_5)\partial_\nu I(\nu,\vec{\Omega}') + \frac{\nu^2}{2}\lambda_5^2\partial_\nu^2 I(\nu,\vec{\Omega}') + \lambda_4 I(\nu,\vec{\Omega}')\Big)d\vec{\Omega}' \\ \qquad + \sigma_s(x)\lambda_1 \displaystyle\int_{S^2}\Big(\nu\lambda_5\partial_\nu I(\nu,\vec{\Omega}') + \lambda_3 I(\nu,\vec{\Omega}')\Big)d\vec{\Omega}' + \sigma_s(x)\lambda_2(< I > -I), \end{cases}$$

and

$$\begin{cases} \bar{Q}^\varepsilon_{s,2}(I) = \sigma_s(x)\displaystyle\int_{S^2}\Big(R_{I,2} + \lambda_3\Big(\nu\lambda_6\partial_\nu I(\nu,\vec{\Omega}') + \frac{\nu^2}{2}\lambda_5^2\partial_\nu^2 I(\nu,\vec{\Omega}') + \varepsilon R_{I,2}\Big)\Big)d\vec{\Omega}' \\ \qquad + \sigma_s(x)\displaystyle\int_{S^2}\Big(\lambda_4\frac{I(\nu',\vec{\Omega}') - I(\nu,\vec{\Omega}')}{\varepsilon} + R_{\frac{\Lambda'}{\Lambda^3},2} I(\nu,\vec{\Omega}')\Big)d\vec{\Omega}' \\ \qquad + \sigma_s(x)\lambda_1\displaystyle\int_{S^2}\Big(\nu\lambda_6\partial_\nu I(\nu,\vec{\Omega}') + \frac{\nu^2}{2}\lambda_5^2\partial_\nu^2 I(\nu,\vec{\Omega}') + \varepsilon R_{I,2}\Big)d\vec{\Omega}' \\ \qquad + \sigma_s(x)\lambda_1\displaystyle\int_{S^2}\Big(\lambda_3\frac{I(\nu',\vec{\Omega}') - I(\nu,\vec{\Omega}')}{\varepsilon} + (\lambda_4 + \varepsilon R_{\frac{\Lambda'}{\Lambda^3},2}) I(\nu,\vec{\Omega}')\Big)d\vec{\Omega}' \\ \qquad + \sigma_s(x)\lambda_2\displaystyle\int_{S^2}\frac{I(\nu',\vec{\Omega}') - I(\nu,\vec{\Omega}')}{\varepsilon}d\vec{\Omega}' \\ \qquad + \sigma_s(x)\lambda_2\displaystyle\int_{S^2}\Big(\frac{\frac{\Lambda'}{\Lambda^3} - 1}{\varepsilon}\Big)I(\nu',\vec{\Omega}')d\vec{\Omega}' + \sigma_s(x)R_{\Lambda,2}\Big(\int_{S^2}\frac{\Lambda'}{\Lambda^3}I(\nu',\vec{\Omega}')d\vec{\Omega}' - I(\nu,\vec{\Omega})\Big), \end{cases} \tag{3.3}$$

where all the coefficients involved in this system are given below. These coefficients come from two different parts: a part of them come from the expansion of the relativistic coefficients and the others come from the Taylor expansion of $I(\nu')$ around the frequency ν. We start with the expansion of the relativistic parameters involve in the expression of the scattering operator: using a Taylor expansion with integral remainder, the coefficient Λ can be written as $\Lambda = 1-\varepsilon\vec{\Omega}.\vec{u}+\varepsilon^2\frac{|\vec{u}|^2}{2}+\varepsilon^3 R_{\Lambda,2}$, which we write $\Lambda = 1+\lambda_1\varepsilon+\lambda_2\varepsilon^2+\varepsilon^3 R_{\Lambda,2}$, with $\lambda_1 = -\vec{\Omega}.\vec{u}$, $\lambda_2 = \frac{|\vec{u}|^2}{2}$ and

$$R_{\Lambda,2} = \frac{1}{\varepsilon^3\sqrt{1-\varepsilon^2|\vec{u}|^2}}\int_{1-\varepsilon^2|\vec{u}|^2}^{1}\frac{1-\varepsilon^2|\vec{u}|^2-s}{4}\frac{1}{s\sqrt{s}}ds + \frac{|\vec{u}|^2}{2\sqrt{1-\varepsilon^2|\vec{u}|^2}}\left(\varepsilon\frac{|\vec{u}|^2}{2}-\vec{\Omega}.\vec{u}\right).$$

In the same way, one has $\frac{\Lambda'}{\Lambda^3} = 1+\varepsilon(3\vec{\Omega}-\vec{\Omega}').\vec{u}+\varepsilon^2 R_{\frac{\Lambda'}{\Lambda^3},2}$, which we write $\frac{\Lambda'}{\Lambda^3} = 1+\lambda_3\varepsilon+\lambda_4\varepsilon^2+\varepsilon^3 R_{\frac{\Lambda'}{\Lambda^3},2}$, with $\lambda_3 = (3\vec{\Omega}-\vec{\Omega}').\vec{u}$, $\lambda_4 = 3(\vec{\Omega},\vec{u})(2\vec{\Omega}-\vec{\Omega}',\vec{u})-|\vec{u}|^2$ and

$$\begin{aligned} R_{\frac{\Lambda'}{\Lambda^3},2} = \frac{1-|\vec{u}|^2}{(1-\varepsilon\vec{\Omega}.\vec{u})^3}\Big(&3\vec{\Omega}'.\vec{u}-3\varepsilon(\vec{\Omega},\vec{u})^2+2\varepsilon^2(\vec{\Omega},\vec{u})^3-3(\vec{\Omega},\vec{u})^2\big(3\vec{\Omega}.\vec{u}-3\varepsilon(\vec{\Omega},\vec{u})^2+\varepsilon^2(\vec{\Omega},\vec{u})^3\big)\Big) \\ &-|\vec{u}|^2\Big(\big(3\vec{\Omega}-\vec{\Omega}',\vec{u}\big)+3\varepsilon\vec{\Omega}.\vec{u}\big(2\vec{\Omega}-\vec{\Omega}',\vec{u}\big)\Big). \end{aligned}$$

Finally, we have $\frac{\Lambda}{\Lambda'} = 1+\varepsilon(\vec{\Omega}'-\vec{\Omega}).\vec{u}+\varepsilon^2\vec{\Omega}'.\vec{u}(\vec{\Omega}'-\vec{\Omega}).\vec{u}+\varepsilon^3 R_{\frac{\Lambda}{\Lambda'},2}$, which we write $\frac{\Lambda}{\Lambda'} = 1+\lambda_5\varepsilon+\lambda_6\varepsilon^2+\varepsilon^3 R_{\frac{\Lambda}{\Lambda'},2}$, with $\lambda_5 = (\vec{\Omega}'-\vec{\Omega}).\vec{u}$, $\lambda_6 = \vec{\Omega}'.\vec{u}(\vec{\Omega}'-\vec{\Omega}).\vec{u}$ and

$$R_{\frac{\Lambda}{\Lambda'},2} = \frac{(\vec{\Omega}',\vec{u})^2(\vec{\Omega}'-\vec{\Omega},\vec{u})}{1-\varepsilon\vec{\Omega}'.\vec{u}}.$$

We now expand the expression of $I(\nu')$ around the frequency ν. Using $\nu'-\nu = (\frac{\Lambda}{\Lambda'}-1)\nu$, a Taylor expansion with integral remainder of I yields

$$I(\nu',\vec{\Omega}') = I(\nu,\vec{\Omega}')+\nu\lambda_5\varepsilon\partial_\nu I(\nu,\vec{\Omega}')+\varepsilon^2\left(\nu\lambda_6\partial_\nu I(\nu,\vec{\Omega}')+\frac{\nu^2}{2}\lambda_5^2\partial_\nu^2 I(\nu,\vec{\Omega}')\right)+\varepsilon^3 R_{I,2},$$

with

$$R_{I,2} = \nu R_{\frac{\Lambda}{\Lambda'},2}\partial_\nu I(\nu,\vec{\Omega}')+\frac{\nu^2}{2\varepsilon^3}\left((\frac{\Lambda}{\Lambda'}-1)^2-(\varepsilon\lambda_5)^2\right)\partial_\nu^2 I(\nu,\vec{\Omega}')+\frac{1}{\varepsilon^3}\int_\nu^{\nu'}\frac{(\nu'-s)^2}{2}\partial_\nu^3 I(s,\vec{\Omega}')ds,$$

and this ends the definition of all the coefficients involved in the expansion at order 2 of the scattering operator.

Expansion of Q_s^ε at order 1

In this part the same expansion of the scattering operator is performed, but we stop at order 1. The method is the same. We write the scattering operator as

$$Q_s^\varepsilon(I) = Q_s^0(I)+\varepsilon Q_s^1(I)+\varepsilon^2\bar{Q}_{s,1}(I),$$

with

$$\left\{\begin{aligned} &Q_s^0(I) = \sigma_s(x)(< I > -I) \\ &Q_s^1(I) = \sigma_s(x)\bigg(\int_{S^2}\Big(\nu\lambda_5\partial_\nu I(\nu,\vec{\Omega}')+\lambda_3 I(\nu,\vec{\Omega}')\Big)d\vec{\Omega}'+\sigma_s(x)\lambda_1(< I > -I)\bigg) \\ &\bar{Q}_{s,1}^\varepsilon(I) = \sigma_s(x,\nu)\int_{S^2}\Big(\lambda_3\lambda_5\nu\partial_\nu I(\nu,\vec{\Omega}')+R_{I,1}+R_{\frac{\Lambda'}{\Lambda^3},1}I(\nu)\Big)d\vec{\Omega}' \\ &\qquad +\sigma_s(x)\lambda_1\int_{S^2}\Big(\frac{I(\nu')-I(\nu)}{\varepsilon}+(\lambda_3+R_{\frac{\Lambda'}{\Lambda^3},1})I(\nu')\Big)d\vec{\Omega}' \\ &\qquad +R_{\Lambda,1}\bigg(\int_{S^2}\frac{\Lambda'}{\Lambda^3}I(\nu',\vec{\Omega}')d\vec{\Omega}'-I(\nu,\vec{\Omega})\bigg). \end{aligned}\right. \tag{3.4}$$

Obviously the terms $Q_s^0(I)$ and $Q_s^1(I)$ are the same than for the expansion at order 2. The only difference with the previous expansions comes from the remainders of the expansions of the coefficients. Again, we have, $\Lambda = 1 - \varepsilon\vec{\Omega}.\vec{u} + \varepsilon^2 R_{\Lambda,1}$, which we write $\Lambda = 1 + \lambda_1\varepsilon + \varepsilon^2 R_{\Lambda,1}$, with

$$\begin{cases} \lambda_1 = -\vec{\Omega}.\vec{u}, \\ R_{\Lambda,1} = \dfrac{1-\sqrt{1-\varepsilon^2|\vec{u}|^2}}{\varepsilon^2\sqrt{1-\varepsilon^2|\vec{u}|^2}}\Big(1-\varepsilon\vec{\Omega}.\vec{u}\Big), \end{cases}$$

We also have $\frac{\Lambda'}{\Lambda^3} = 1 + \varepsilon(3\vec{\Omega}-\vec{\Omega}').\vec{u} + \varepsilon^2 R_{\frac{\Lambda'}{\Lambda^3},1}$, which we write $\frac{\Lambda'}{\Lambda^3} = 1 + \lambda_3\varepsilon + \varepsilon^2 R_{\frac{\Lambda'}{\Lambda^3},1}$, with

$$\begin{cases} \lambda_3 = (3\vec{\Omega}-\vec{\Omega}').\vec{u}, \\ R_{\frac{\Lambda'}{\Lambda^3},1} = \dfrac{(3\vec{\Omega}-\vec{\Omega}',\vec{u})}{\varepsilon}\dfrac{1-(1-\varepsilon\vec{\Omega},\vec{u})^3}{(1-\varepsilon\vec{\Omega},\vec{u})^3} - \dfrac{\big((\vec{\Omega},\vec{u})^2-(\vec{\Omega},\vec{u})^3\big)\big(1-|\vec{u}|^2\big)}{(1-\varepsilon\vec{\Omega},\vec{u})^3}. \end{cases}$$

Finally, we have $\frac{\Lambda}{\Lambda'} = 1 + \varepsilon(\vec{\Omega}'-\vec{\Omega}).\vec{u} + \varepsilon^2 R_{\frac{\Lambda}{\Lambda'},1}$, which we write $\frac{\Lambda}{\Lambda'} = 1 + \lambda_5\varepsilon + \varepsilon^2 R_{\frac{\Lambda}{\Lambda'},1}$, with

$$\begin{cases} \lambda_5 = (\vec{\Omega}'-\vec{\Omega}).\vec{u}, \\ R_{\frac{\Lambda}{\Lambda'},1} = (\vec{\Omega}',\vec{u})\dfrac{(\vec{\Omega}'-\vec{\Omega},\vec{u})}{1-\varepsilon\vec{\Omega}'.\vec{u}}. \end{cases}$$

We now make a Taylor expansion, with respect to ν, of I :

$$I(\nu',\vec{\Omega}') = I(\nu,\vec{\Omega}') + \nu\lambda_5\varepsilon\partial_\nu I(\nu,\vec{\Omega}') + \varepsilon^2 R_{I,1},$$

with $R_{I,1} = \nu R_{\frac{\Lambda}{\Lambda'},1}\partial_\nu I + \frac{1}{\varepsilon^2}\int_\nu^{\nu'}(\nu'-s)\partial_\nu^2 I(s)ds$.

Expansion of Q_s^ε at order 0

In this part we make the same development of the scattering operator but we stop at order 1. The method is the same. We write the scattering operator as

$$Q_s^\varepsilon(I) = Q_s^0(I) + \varepsilon\bar{Q}_{s,0}(I).$$

with

$$\bar{Q}_{s,0}^\varepsilon(I) = \sigma_s(x)\int_{S^2}\big(R_{I,0} + R_{\frac{\Lambda'}{\Lambda^3},0}I(\nu',\vec{\Omega}')\big)d\vec{\Omega}' + R_{\Lambda,0}\bigg(\int_{S^2}\frac{\Lambda'}{\Lambda^3}I(\nu',\vec{\Omega}')d\vec{\Omega}' - I(\nu,\vec{\Omega})\bigg). \tag{3.5}$$

Obviously the term $Q_s^0(I)$ is the same than for the expansion at order 2. Once again, the only difference comes from the remainders. We have $\Lambda = 1 + \varepsilon R_{\Lambda,0}$, with $R_{\Lambda,0} = \frac{1-\varepsilon\vec{\Omega}.\vec{u}-\sqrt{1-\varepsilon^2|\vec{u}|^2}}{\varepsilon\sqrt{1-\varepsilon^2|\vec{u}|^2}}$. We also have $\frac{\Lambda'}{\Lambda^3} = 1 + \varepsilon R_{\frac{\Lambda'}{\Lambda^3},0}$, with $R_{\frac{\Lambda'}{\Lambda^3},0} = \frac{1-\varepsilon^2|\vec{u}|^2}{\varepsilon}\frac{1-\varepsilon\vec{\Omega}.\vec{u}-(1-\varepsilon\vec{\Omega}.\vec{u})^3}{(1-\varepsilon\vec{\Omega}.\vec{u})^3}$. Finally, we have $\frac{\Lambda}{\Lambda'} = 1 + \varepsilon R_{\frac{\Lambda}{\Lambda'},0}$, with $R_{\frac{\Lambda}{\Lambda'},0} = \frac{(\vec{\Omega}'-\vec{\Omega},\vec{u})}{1-\varepsilon\vec{\Omega}'.\vec{u}}$. We make a Taylor expansion, with respect to ν, of I :

$$I(\nu',\vec{\Omega}') = I(\nu,\vec{\Omega}') + \varepsilon R_{I,0},$$

with $R_{I,0} = \frac{1}{\varepsilon}\int_\nu^{\nu'}\partial_\nu I(s)ds$.

Expansion of the emission absorption operator

As for the scattering operator, a Hilbert expansion with exact residual term of the emission absorption operator is performed. The study is much simpler than for the scattering operator since the scaling is less severe. We recall here the definition of the emission absorption operator: given two function $T : [0,T^f] \times \mathbb{R}^3_x \to \mathbb{R}^+$ and $I : [0,T^f] \times \mathbb{R}^3_x \times \mathbb{R}^+_\nu \times S^2 \to \mathbb{R}^+$, it is defined by

$$Q_a^\varepsilon(I,T) = \frac{\sigma_a(\nu_0^\varepsilon)}{\Lambda^{\varepsilon 2}} \Big(B(\nu_0^\varepsilon, T) - (\Lambda^\varepsilon)^3 I \Big). \tag{3.6}$$

Dropping the ε in the coefficients Λ^ε and ν_0^ε for ease of notations, we write

$$Q_a^\varepsilon(I,T) = Q_a^0(I,T) + \varepsilon \bar{Q}_a^\varepsilon(I,T), \tag{3.7}$$

with

$$\begin{cases} Q_a^0(I,T) = \sigma_a(\nu)\big(B(\nu,T) - I\big), \\ \bar{Q}_a^\varepsilon(I,T) = \dfrac{1-\Lambda^2}{\varepsilon\Lambda^2}\sigma_a(\nu_0)\big(B(\nu_0,T) - I_0\big) + \dfrac{\sigma_a(\nu_0) - \sigma_a(\nu)}{\varepsilon}\big(B(\nu_0,T) - I\big) \\ \qquad + \sigma_a(\nu)\dfrac{B(\nu_0,T) - B(\nu,T)}{\varepsilon} + \sigma_a(\nu_0)\dfrac{I - I_0}{\varepsilon}. \end{cases} \tag{3.8}$$

3.1.2 Second step: Formal Hilbert expansion of the transport equation

In this part we find the formal asymptotic limit of the relativistic transfer equation in the non-equilibrium regime using a formal Hilbert expansion, that is

$$\begin{cases} I^\varepsilon = I^0 + \varepsilon I^1 + \varepsilon^2 I^2 + \mathscr{O}(\varepsilon^3), \\ T^\varepsilon = T^0 + \mathscr{O}(\varepsilon). \end{cases} \tag{3.9}$$

The limit will only be formal, in the sense that the remainders in (3.9) are not explicitly bounded in some norm. This will be performed in the next section. Take care to the fact that in the notations, the subscript 0 refers to quantities computed in the moving frame, while the power 0 refers to the first order term in the expansion in power of ε.

The choice of the scaling is driven by the fact that the temperature T is only involved in $\mathscr{O}(1)$ terms in (3.1), while I is involved in $\mathscr{O}(\varepsilon^{-2})$ terms. Since the scattering operator Q_s is linear, one has $Q_s^\varepsilon(I^\varepsilon) = Q_s^\varepsilon(I^0) + \varepsilon Q_s^\varepsilon(I^1) + \varepsilon^2 Q_s^\varepsilon(I^2) + \mathscr{O}(\varepsilon^3)$. We use the expansion at order 2 of the scattering operator for the zero-th order term I^0, the expansion at order 1 for the first order term I^1 and the expansion at order 0 for the second order term I^2. It yields

$$\begin{cases} Q_s^\varepsilon(I^0) = Q_s^0(I^0) + \varepsilon Q_s^1(I^0) + \varepsilon^2 Q_s^2(I^0) + \varepsilon^3 \bar{Q}_{s,2}^\varepsilon(I^0), \\ \\ Q_s^\varepsilon(I^1) = Q_s^0(I^1) + \varepsilon Q_s^1(I^1) + \varepsilon^2 \bar{Q}_{s,1}^\varepsilon(I^1), \\ \\ Q_s^\varepsilon(I^2) = Q_s^0(I^2) + \varepsilon \bar{Q}_{s,0}^\varepsilon(I^2). \end{cases}$$

The previous expansion (3.7) of the emission absorption operator yields $Q_a^\varepsilon(I^\varepsilon,T^\varepsilon) = Q_a^0(I^\varepsilon,T^\varepsilon) + \bar{Q}_a^\varepsilon(I^\varepsilon,T^\varepsilon)$. Moreover, the expansions (3.9) of the unknowns I^ε and T^ε and a Taylor expansion of the Planck function $B(\nu,T^\varepsilon)$ formally leads to $Q_a^0(I^\varepsilon,T^\varepsilon) = Q_a^0(I^0,T^0) + \mathscr{O}(\varepsilon)$. It yields, taking into account all the remainders $\bar{Q}_{s,i}$, $i = 0,1,2$ and $\bar{Q}_a$ as $\mathscr{O}(\varepsilon)$ terms,

$$\begin{cases} \partial_t(I^0 + \varepsilon I^1 + \varepsilon^2 I^2) + \dfrac{1}{\varepsilon}\vec{\Omega}.\nabla_x(I^0 + \varepsilon I^1 + \varepsilon^2 I^2) = \dfrac{Q_s^0(I^0)}{\varepsilon^2} + \dfrac{Q_s^1(I^0) + Q_s^0(I^1)}{\varepsilon} \\ \qquad\qquad + Q_s^2(I^0) + Q_s^1(I^1) + Q_s^0(I^2) + Q_a^0(I^0,T^0) + \mathscr{O}(\varepsilon), \\ \\ \partial_t T^0 + \nabla.(T^0\vec{u}) + \Gamma T^0 \nabla.\vec{u} = -\displaystyle\int_{\nu,\Omega} Q_a^0(I^0,T^0) + \mathscr{O}(\varepsilon). \end{cases}$$

We now study all the terms with the same power of ε. In the forthcoming computations, the formula $\int_\Omega (\vec{\Omega}, \vec{u})^2 = \frac{|\vec{u}|^2}{3}$ will be often used. First, in $\frac{1}{\varepsilon^2}$, one has $Q_s^0(I^0) = 0$, that is $\sigma_s(x)(I^0 - \int_\Omega I^0) = 0$, and thus I^0 is independent of the angular direction $\vec{\Omega}$. At the order $\frac{1}{\varepsilon}$, one has $\vec{\Omega}.\nabla_x I^0 = Q_s^0(I^1) + Q_s^1(I^0)$, which yields

$$\vec{\Omega}.\nabla_x I^0 = \sigma_s(x)\bigg(\int_{\Omega'} I^1(\vec{\Omega}') - I^1\bigg) + \sigma_s(x)\bigg(\int_{\Omega'} \lambda_5 \nu \partial_\nu I^0 + \int_{\Omega'} \lambda_3 I^0\bigg).$$

Using the relations $\int_{\Omega'} \lambda_5 = -\vec{\Omega}.\vec{u}$ and $\int_{\Omega'} \lambda_3 = 3\vec{\Omega}.\vec{u}$, one finds

$$\sigma_s(x)\bigg(I^1 - \int_{\Omega'} I^1(\vec{\Omega}')\bigg) = -\vec{\Omega}.\nabla_x I^0 - \sigma_s(x)\vec{\Omega}.\vec{u}\nu\partial_\nu I^0 + 3\sigma_s(x)\vec{\Omega}.\vec{u} I^0 \tag{3.10}$$

This kind of equation is studied in [AG12]. Let us recall the main lines. We introduce $\mathscr{K} : \phi \mapsto \int_\Omega \phi$, which is an Hilbert Schmidt operator, and we study the auxiliary equation

$$\begin{cases} (I_d - \mathscr{K}) b_j(\vec{\Omega}) = \Omega_j, \\ \displaystyle\int_\Omega b_j = 0. \end{cases}$$

Since $\int_\Omega \Omega_j = \big(\int_\Omega \vec{\Omega}\big)_j = 0$, the Fredholm theory gives the existence of a solution I_1. In particular, it can be shown, see [AG12], that there exists a unique solution $b_j(\vec{\Omega}) \in \mathrm{Ker}(I_d - \mathscr{K})^\perp$. The solutions of equation (3.10) are the functions of the form

$$I^1 = -\frac{1}{\sigma_s(x)}\vec{\Omega}.\nabla_x I^0 - \vec{\Omega}.\vec{u}\nu\partial_\nu I^0 + 3\vec{\Omega}.\vec{u} I^0 + C^1(t,x,\nu), \tag{3.11}$$

where $C^1(t,x,\nu)$, constant in $\vec{\Omega}$, is an arbitrary solution of the homogeneous equation $(\mathscr{K} - I_d)C_1 = 0$. Finally, at the order 0, one has

$$\begin{cases} \partial_t I^0 + \vec{\Omega}.\nabla_x I^1 = Q_s^0(I^2) + Q_s^1(I^1) + Q_s^2(I^0) + Q_a^0(I^0, T^0), \\ \partial_t T^0 + \nabla.(T^0 \vec{u}) + \Gamma T^0 \nabla.\vec{u} = -\displaystyle\int_\nu Q_a^0(I^0, T^0). \end{cases} \tag{3.12}$$

We have, using the definitions of the λ_i (part 3.1.1) and I^1,

$$\begin{aligned} Q_s^1(I^1) = &-\nu \int_\Omega (\vec{\Omega}, \vec{u})(\vec{\Omega}, \nabla \partial_\nu I^0) - \sigma_s(x)\frac{|\vec{u}|^2}{3}\big(\nu\partial_\nu I^0 + \nu^2 \partial_\nu^2 I^0\big) + \sigma_s(x)|\vec{u}|^2 \nu \partial_\nu I^0 \\ &+ \int_\Omega (\vec{\Omega}, \vec{u})(\vec{\Omega}, \nabla I^0) + \sigma_s(x)\frac{|\vec{u}|^2}{3}\nu\partial_\nu I^0 - \sigma_s(x)|\vec{u}|^2 I^0 - (\vec{\Omega}, \vec{u})(\vec{\Omega}, \nabla I^0) \\ &- \sigma_s(x)(\vec{\Omega}, \vec{u})^2 \nu \partial_\nu I^0 + 3\sigma_s(\vec{\Omega}, \vec{u})^2 I^0 + \sigma_s(x)(\vec{\Omega}, \vec{u})(3C^1 - \nu\partial_\nu C^1). \end{aligned}$$

In the same way, one has

$$Q_s^2(I^0) = -2(\vec{\Omega}, \vec{u})^2 \sigma_s(x)\nu\partial_\nu I^0 + \sigma_s(x)\frac{\nu^2}{2}\partial_\nu^2 I^0\bigg(\frac{|\vec{u}|^2}{3} + (\vec{\Omega}, \vec{u})^2\bigg) + \sigma_s(x) I^0\bigg(3(\vec{\Omega}, \vec{u})^2 - |\vec{u}|^2\bigg),$$

and $Q_s^0(I^2) = \sigma_s(x)\big(\int_\Omega I^2 - I^2\big)$. We thus have

$$\begin{cases} \partial_t I^0 + \vec{\Omega}.\nabla_x I^1 = -\nu \displaystyle\int_\Omega (\vec{\Omega},\vec{u})(\vec{\Omega},\nabla\partial_\nu I^0) + \sigma_s(x)\nu\partial_\nu I^0\Big(|\vec{u}|^2 - 3(\vec{\Omega},\vec{u})^2\Big) \\ \qquad\qquad + \sigma_s(x)\nu^2\partial_\nu^2 I^0\Big((\vec{\Omega},\vec{u})^2 - \dfrac{|\vec{u}|^2}{3}\Big) + \displaystyle\int_\Omega (\vec{\Omega},\vec{u})(\vec{\Omega},\nabla I^0) - (\vec{\Omega},\vec{u})(\vec{\Omega},\nabla I^0) \\ \qquad\qquad + 2\sigma_s(x)I^0\Big(3(\vec{\Omega},\vec{u})^2 - |\vec{u}|^2\Big) + Q_a^0(I^0,T^0) + \sigma_s(x)(\vec{\Omega},\vec{u})(3C^1 - \nu\partial_\nu C^1) \\ \qquad\qquad + \sigma_s(x)\Big(\displaystyle\int_\Omega I^2 - I^2\Big), \\ \partial_t T^0 + \nabla.(T^0\vec{u}) + \Gamma T^0\nabla.\vec{u} = -\displaystyle\int_\nu Q_a^0(I^0,T^0). \end{cases} \tag{3.13}$$

The first equation can be rewritten $\sigma_s(x)(\mathscr{K} - I_d)I^2 = \partial_t I^0 - g$, with an obvious definition of the function g. Using once again the Fredholm theory, this equation has a solution if and only if the compatibility condition $\partial_t I^0 - g \in \mathrm{Ker}(\mathscr{K} - I_d)^\perp$, i.e. $\int_\Omega(\partial_t I^0 - g) = \partial_t I^0 - \int_\Omega g = 0$ is satisfied. This gives us $\partial_t I^0 = \int_\Omega g$ and thus I^2 satisfy $\sigma_s(x)(\mathscr{K} - I_d)I_2 = \int_\Omega g - g$. Using the same arguments than for the computation of I_1, the solution of this equation is of the form

$$\begin{aligned} I^2 &= \Big((\vec{\Omega},\vec{u})^2 - \frac{|\vec{u}|^2}{3}\Big)\Big(6I^0 - 3\nu\partial_\nu I^0 + \nu^2\partial_\nu^2 I^0\Big) + \frac{1}{\sigma_s^2(x)}\sum_{i,j}\Big(\vec{\Omega}_i\vec{\Omega}_j - \int_\Omega \vec{\Omega}_i\vec{\Omega}_j\Big)\partial_{x_j}\partial_{x_i} I^0 \\ &+ \frac{1}{\sigma_s(x)}\sum_{i,j}\Big(\vec{\Omega}_i\vec{\Omega}_j - \int_\Omega \vec{\Omega}_i\vec{\Omega}_j\Big)\Big((\nu\partial_\nu I^0 - 3I^0)\partial_{x_j}u_i + \big(\nu\partial_\nu\partial_{x_j} I^0 - 3\partial_{x_j} I^0\big)u_i\Big) \\ &- \sum_{i,j}\Big(\vec{\Omega}_i\vec{\Omega}_j - \int_\Omega \vec{\Omega}_i\vec{\Omega}_j\Big)u_i\partial_{x_j} I^0 + (\vec{\Omega},\vec{u})\big(3C^1 - \nu\partial_\nu C1\big) - \frac{1}{\sigma_s(x)}(\vec{\Omega},\nabla C^1) + C^2(t,x,\nu), \end{aligned} \tag{3.14}$$

where, $C^2(t,x,\nu)$ is an arbitrary solution of the homogeneous equation $(\mathscr{K} - I_d)C_2 = 0$. Setting $\rho = I^0$ and $\bar{T} = T^0$, integrating the first equation of (3.13) on S^2 and computing all the terms, we find

$$\begin{cases} \partial_t\rho - \nabla.\Big(\dfrac{\nabla\rho}{3\sigma_s(x)}\Big) + \nabla.\big(\rho\vec{u}\big) = \dfrac{\nabla.\vec{u}}{3}\nu\partial_\nu\rho + \sigma_a(\nu)(B(\nu,\bar{T}) - \rho) + \mathscr{O}(\varepsilon), \\ \partial_t\bar{T} + \nabla.(\bar{T}\vec{u}) + \Gamma\bar{T}\nabla.\vec{u} = -\displaystyle\int_\nu \sigma_a(\nu)(B(\nu,\bar{T}) - \rho) + \mathscr{O}(\varepsilon). \end{cases}$$

Finally, dropping formally all the $\mathscr{O}(\varepsilon)$ terms, we find the following drift-diffusion equation

$$\begin{cases} \partial_t\rho - \nabla.\Big(\dfrac{\nabla\rho}{3\sigma_s(x)}\Big) + \nabla.\big(\rho\vec{u}\big) = \dfrac{\nabla.\vec{u}}{3}\nu\partial_\nu\rho + \sigma_a(\nu)(B(\nu,\bar{T}) - \rho), \\ \partial_t\bar{T} + \nabla.(\bar{T}\vec{u}) + \Gamma\bar{T}\nabla.\vec{u} = -\displaystyle\int_\nu \sigma_a(\nu)(B(\nu,\bar{T}) - \rho), \end{cases} \tag{3.15}$$

which ends the proof of lemma 3.1. □

3.2 A rigorous proof of convergence

This part is devoted to the proof of convergence of the solution $(I^\varepsilon, T^\varepsilon)$ of the relativistic transfer equations (3.1) to the solution of the drift diffusion system (3.2) as $\varepsilon \to 0$. On the contrary to the previous section, in which the limit was obtain formally (lemma 3.1), the remainders of the source terms expansions (part 3.1.1) are shown to be bounded in $L_t^\infty(L_{x,\nu}^2)$ (lemmas 3.5 and 3.6). The main result (theorem 3.4) deals with the proof of strong convergence in L^2 of the difference between the solution of the transfer equations (3.1) and the solution of the drift diffusion system (3.2).

In a first part are introduced several new assumptions, which are different from the assumptions H done in the previous section, due to the fact that we need more regularity on the coefficients σ_a, σ_s and $\vec{u}$. Several a priori estimates are proved, such as a maximum principle for the drift diffusion system (3.2) (lemma 3.2) and some regularity results for the solution of this system (lemma 3.19), and we introduce our main result (theorem 3.4). The second part deals with the proof of this result, and is divided in three parts, due to the technical aspects of this proof.

3.2.1 A priori estimates and main result

We assume more regularity on the parameters $\vec{u}$, σ_s and σ_a than in the previous section. This is summarized here, where the assumptions ($\overline{\text{H1}}$)$-$($\overline{\text{H4}}$) are related to the regularity of the coefficients σ_a, σ_s,$\vec{u}$ and on the parameter ε, the assumption ($\overline{\text{H5}}$) (respectively the assumption ($\overline{\text{H6}}$)) is related to the initial conditions (respectively to the boundary conditions) of the solution of the drift diffusion system (3.2).

- ($\overline{\text{H1}}$) The velocity field satisfies $\vec{u} \in W^{4,\infty}([0,T^f]\times\mathbb{R}^3_x)$. Moreover, $u^* = \|\vec{u}\|_{L^\infty}$ is such that $u^* < c$.
- ($\overline{\text{H2}}$) The coefficients ε and u^* are such that there exists $\varepsilon^* < 1$ such that $u^*\varepsilon \leq \varepsilon^*$. It yields the positivity of $1-\varepsilon^2|\vec{u}|^2$ involved in the expression of the Lorentz coefficient γ^ε.
- ($\overline{\text{H3}}$) Smoothness of the scattering coefficient: $\sigma_s \in W^{3,\infty}(\mathbb{R}^3_x)$ and $\sigma_s > 0$.
- ($\overline{\text{H4}}$) Smoothness of the emission absorption coefficient: $\sigma_a \in W^{3,\infty}(\mathbb{R}^+_\nu)\cap L^2(\mathbb{R}^+_\nu)$ and $\sigma_a > 0$.
- ($\overline{\text{H5}}$) The initial conditions $(\rho^{in},\bar{T}^{in})$ of the drift diffusion system (3.2) are such that $\nu^k\partial^k_\nu\partial^m_t\partial^l_{x_j}\rho^{in} \in L^\infty\big([0,T^f];L^2(\mathbb{R}^3_x\times\mathbb{R}^+_\nu)\big)$, $k,l,j\in(0,1,2,3), m\in(0,1)$ and $\partial^k_{x_j}\partial^m_t\bar{T}^{in} \in L^\infty\big(([0,T^f];L^2(\mathbb{R}^3_x)\big)$, $k,j\in(0,1,2,3), m\in(0,1)$. Moreover, there exists two bounded and positive constants $\bar{T}_*$ and $\bar{T}^*$ such that $\forall(x,\nu)\in\mathbb{R}^3_x\times\mathbb{R}^+_\nu$, $\bar{T}_*\leq\bar{T}^{in}(x)\leq\bar{T}^*$ and $0<B(\nu,\bar{T}_*)\leq\rho^{in}(x,\nu)\leq B(\nu,\bar{T}^*)$.
- ($\overline{\text{H6}}$) The behavior at the infinity of the solution of the drift diffusion system (3.2) is such that $\lim\limits_{|x_j|\to\infty}\partial^k_{x_j}\partial^m_t\bar{T}=0$, $k\in(0,1,2,3)$, $j\in(1,2,3)$ and $m\in(0,1)$ and $\lim\limits_{|x_j|\to\infty}\nu^k\partial^k_\nu\partial^m_t\partial^l_x\rho = \lim\limits_{\nu\to0}\nu^k\partial^k_\nu\partial^m_t\partial^l_x\rho = \lim\limits_{\nu\to\infty}\nu^k\partial^k_\nu\partial^m_t\partial^l_x\rho = 0$, $k,l\in(0,1,2,3), j\in(1,2,3)$ and $m\in(0,1)$, which means that the solution of the drift diffusion system and its derivatives have a strongly decaying behavior at $|x|\to\infty$, $\nu\to0$ and $\nu\to\infty$.

The assumption $\overline{\text{H2}}$ yields in particular the positivity of the Lorentz factor γ^ε. Moreover, this assumption together with the assumption $\overline{\text{H1}}$ lead to an equivalent of the estimates (2.1) for the relativistic coefficient Λ^ε

$$0<\Lambda_*\leq\Lambda^\varepsilon(t,x,\vec{\Omega})\leq\Lambda^*,\qquad \forall(t,x,\vec{\Omega})\in[0,T^f]\times\mathbb{R}^3_x\times S^2, \tag{3.16}$$

where the notation Λ_* and Λ^* have been kept for simplicity. The assumption $\overline{\text{H4}}$ which stipulates that the emission absorption coefficient satisfies $\sigma_a\in L^2(\mathbb{R}^+_\nu)$ is purely technical, in the sense that it has no physical meaning, but is necessary to prove the convergence result in L^2 norm, since this result needs the derivatives of the solution of the drift diffusion system to belong to L^2 (B.1,B.3). The end of this section deals with a priori estimates and with our main result. First (theorem (3.2)), a minimum maximum principle is proved for the solution of the drift diffusion system (3.2), using the same tools than for the radiative transfer equations (theorem 2.1) together with a trick to treat the diffusion term. Secondly (lemma 3.3), a regularity result is provided for the solution of the drift diffusion system (3.2), whose proof is postponed to the appendix. Finally (theorem 3.4), the convergence result of the solution of the radiative transfer equations (3.1) to the solution of the drift diffusion system (3.2) as $\varepsilon\to0$ is introduced.

Lemma 3.2 (Maximum Principle). *Assume that hypotheses* $(\overline{H1})$*,*$(\overline{H3})$*,*$(\overline{H4})$ *and* $(\overline{H5})$ *are satisfied. Then* $\forall(t,x,\nu)\in[0,T^f]\times\mathbb{R}^3_x\times\mathbb{R}^+_\nu$*, the solution of the drift diffusion system* (3.2) *satisfies the a priori estimates* $\bar{T}_*(t)\le\bar{T}(t,x)\le\bar{T}^*(t)$ *and* $B(\nu,\bar{T}_*(t))\le\rho(t,x,\nu)\le B(\nu,\bar{T}^*(t))$*, with*

$$\begin{cases}\bar{T}_*(t)=\bar{T}_*e^{-t(\Gamma+\frac{4}{3})\|\nabla.\vec{u}\|_{L^\infty_{t,x}}},\\ \bar{T}^*(t)=\bar{T}^*e^{t(\Gamma+\frac{4}{3})\|\nabla.\vec{u}\|_{L^\infty_{t,x}}},\end{cases}\tag{3.17}$$

where the constants $\bar{T}_*$ *and* $\bar{T}^*$ *are defined in assumption* $(\overline{H5})$*.*

Proof. Since the arguments are similar, we only show the proof for the maximum principle. We denote $B^*_\nu=B(\nu,\bar{T}^*)$ for ease of notations. The proof is mainly the same than for the relativistic transfer equations (1.1), except that we need to treat the second order derivative. To achieve this we use a method of Carrillo et al [CRS08], which is to introduce a function sgn^+_α, where $0<\alpha\le 1$ is a small parameter, as a non decreasing regularization of the sgn^+ function defined in (1.12). It yields in particular, using an integration by parts, $\int_x \text{sgn}^+_\alpha\big(\rho-B^*_\nu\big)\nabla.\big(\frac{\nabla\rho}{3\sigma_s}\big)=-\int_x\big(\text{sgn}^+_\alpha\big)'\big(\rho-B^*_\nu\big)|\nabla\rho|^2/3\sigma_s$, due to the fact that $\bar{T}^*$ does not depend on the space variable x. Multiplying the first equation of (3.2) by $\text{sgn}^+_\alpha\big(\rho-B^*_\nu\big)$ and integrating on $\mathbb{R}^3_x\times\mathbb{R}^+_\nu$ yields, with an integration by parts,

$$\begin{aligned}&\int_{x,\nu}\text{sgn}^+_\alpha\big(\rho-B^*_\nu\big)\partial_t\rho+\int_{x,\nu}\big(\text{sgn}^+_\alpha\big)'\big(\rho-B^*_\nu\big)\frac{|\nabla\rho|^2}{3\sigma_s(x)}+\int_{x,\nu}\text{sgn}^+_\alpha\big(\rho-B^*_\nu\big)\nabla.\big(\rho\vec{u}\big)\\&=\int_{x,\nu}\text{sgn}^+_\alpha\big(\rho-B^*_\nu\big)\Big(\frac{\nabla.\vec{u}}{3}\nu\partial_\nu\rho+\sigma_a(\nu)(B(\nu,\bar{T})-\rho)\Big).\end{aligned}$$

The positivity of the scattering coefficient σ_s (assumption $(\overline{H3})$) yields

$$\begin{aligned}&\int_{x,\nu}\text{sgn}^+_\alpha\big(\rho-B^*_\nu)\big)\partial_t\rho+\int_{x,\nu}\text{sgn}^+_\alpha\big(\rho-B^*_\nu\big)\nabla.\big(\rho\vec{u}\big)\\&\le\int_{x,\nu}\text{sgn}^+_\alpha\big(\rho-B^*_\nu)\big)\Big(\frac{\nabla.\vec{u}}{3}\nu\partial_\nu\rho+\sigma_a(\nu)(B(\nu,\bar{T})-\rho)\Big).\end{aligned}$$

We now pass to the limit as $\alpha\to 0$ in this inequality, where only the sgn^+_α function depends on α. It yields

$$\begin{aligned}&\int_{x,\nu}\text{sgn}^+\big(\rho-B^*_\nu\big)\partial_t\rho+\int_{x,\nu}\text{sgn}^+\big(\rho-B^*_\nu\big)\nabla.\big(\rho\vec{u}\big)\\&\le\int_{x,\nu}\text{sgn}^+\big(\rho-B^*_\nu\big)\Big(\frac{\nabla.\vec{u}}{3}\nu\partial_\nu\rho+\sigma_a(\nu)(B(\nu,\bar{T})-\rho)\Big).\end{aligned}$$

Like for the proof of the minimum-maximum principle for the transfer equations (theorem 2.1), we write an inequality satisfied by $\int_{x,\nu}\big(\rho-B^*_\nu\big)^+$. One has $\int_{x,\nu}\text{sgn}^+\big(\rho-B^*_\nu\big)\nabla.\big(\rho\vec{u}\big)=\int_{x,\nu}\nabla.\big((\rho-B^*_\nu)^+\vec{u}\big)+\int_{x,\nu}\text{sgn}^+\big(\rho-B^*_\nu\big)B^*_\nu\nabla.\vec{u}=\int_{x,\nu}\text{sgn}^+\big(\rho-B^*_\nu\big)B^*_\nu\nabla.\vec{u}$. In the same way, one has $\int_{x,\nu}\text{sgn}^+\big(\rho-B^*_\nu\big)\partial_t\rho=\frac{d}{dt}\int_{x,\nu}\big(\rho-B^*_\nu\big)^++\int_{x,\nu}\text{sgn}^+\big(\rho-B^*_\nu\big)\partial_t B^*_\nu$ and $\int_{x,\nu}\text{sgn}^+\big(\rho-B^*_\nu\big)\frac{\nabla.\vec{u}}{3}\nu\partial_\nu\rho=\int_{x,\nu}\frac{\nabla.\vec{u}}{3}\nu\partial_\nu\big(\rho-B^*_\nu\big)^++\int_{x,\nu}\text{sgn}^+\big(\rho-B^*_\nu\big)\frac{\nabla.\vec{u}}{3}\nu\partial_\nu B^*_\nu$. An integration by parts yields $\int_{x,\nu}\text{sgn}^+\big(\rho-B^*_\nu\big)\frac{\nabla.\vec{u}}{3}\nu\partial_\nu\rho=-\int_{x,\nu}\frac{\nabla.\vec{u}}{3}\big(\rho-B^*_\nu\big)^++\int_{x,\nu}\text{sgn}^+\big(\rho-B^*_\nu\big)\frac{\nabla.\vec{u}}{3}\nu\partial_\nu B^*_\nu$. This gives us

$$\begin{aligned}\frac{d}{dt}\int_{x,\nu}\big(\rho-B^*_\nu\big)^+\le&-\int_{x,\nu}\frac{\nabla.\vec{u}}{3}\big(\rho-B^*_\nu\big)^++\int_{x,\nu}\sigma_a(\nu)(B(\nu,\bar{T})-\rho)\text{sgn}^+\big(\rho-B^*_\nu\big)\\&-\int_{x,\nu}\text{sgn}^+\big(\rho-B^*_\nu\big)\Big(\partial_t B^*_\nu+B^*_\nu\nabla.\vec{u}-\frac{\nabla.\vec{u}}{3}\nu\partial_\nu B^*_\nu\Big).\end{aligned}$$

We write the equation satisfied by $\int_{\mathbb{R}^3_x}\big(T-T^*)\big)^+$, where T is the solution of the drift diffusion equation (3.2). One finds the same equation (see equation (2.6)) than for the relativistic transfer equations

$$\frac{d}{dt}\int_x (\bar T-\bar T^*)^+ + \Gamma\int_x (\bar T-\bar T^*)^+\nabla.\vec u + \int_x \mathrm{sgn}^+(\bar T-\bar T^*)\Big(\partial_t \bar T^* + (\Gamma+1)\bar T^*\nabla.\vec u\Big)$$
$$= -\int_x \mathrm{sgn}^+(\bar T-\bar T^*)\sigma_a(\nu)(B(\nu,\bar T)-\rho).$$

Once again, one has $\int_x \big(\mathrm{sgn}^+(\rho - B(\nu,\bar T^*)) - \mathrm{sgn}^+(\bar T-\bar T^*)\big)\sigma_a(\nu)(B(\nu,\bar T)-\rho)\le 0$ which is non negative thanks to the positivity of the emission absorption coefficient (assumption ($\overline{H4}$)). Introducing $H(t)=\int_{x,\nu}\big(\rho-B(\nu,\bar T^*)\big)^+ + \int_x \big(\bar T-\bar T^*)\big)^+$, one finds

$$H'(t)\le\Big(\Gamma+\frac{1}{3}\Big)\|\nabla.\vec u\|_{L^\infty_x}H(t) - \int_x \mathrm{sgn}^+(\bar T-\bar T^*)\Big(\partial_t\bar T^*+(\Gamma+1)\bar T^*\nabla.\vec u\Big)$$
$$-\int_{x,\nu}\mathrm{sgn}^+\big(\rho-B^*_\nu\big)\Big(\partial_t B^*_\nu+\nabla.\vec u B^*_\nu - \frac{\nabla.\vec u}{3}\nu\partial_\nu B^*_\nu\Big).$$

The key of the proof is that by definition of $\bar T^*$, one has

$$\begin{cases}\displaystyle\int_x \mathrm{sgn}^+(\bar T-\bar T^*)\Big(\partial_t\bar T^*+(\Gamma+1)\bar T^*\nabla.\vec u\Big)\ge 0,\\ \displaystyle\int_{x,\nu}\mathrm{sgn}^+\big(\rho-B^*_\nu\big)\Big(\partial_t B^*_\nu+\nabla.\vec u B^*_\nu-\frac{\nabla.\vec u}{3}\nu\partial_\nu B^*_\nu\Big)\ge 0,\end{cases}\tag{3.18}$$

Actually, the definition of $\bar T^*(t)$ (3.17) yields $\partial_t\bar T^*(t)/\bar T^*(t)=(\Gamma+\frac{4}{3})\|\nabla.\vec u\|_{L^\infty_{t,x}}$. It gives the first line of (3.18). For the second line, the definition of the Planck function yields

$$\partial_t B^*_\nu+\nabla.\vec u B^*_\nu-\frac{\nabla.\vec u}{3}\nu\partial_\nu B^*_\nu=\frac{\nu}{\bar T^*}\frac{B^*_\nu}{1-e^{-\nu/\bar T^*}}\Big(\frac{\partial_t\bar T^*}{\bar T^*}+\frac{\nabla.\vec u}{3}\Big),$$

which is non negative by definition of $\bar T^*$ (3.17). It yields $H'(t)\le\big(\Gamma+\frac{1}{3}\big)\|\nabla.\vec u\|_{L^\infty_x}H(t)$. Since $H(0)=0$ (assumption ($\overline{H5}$)), the Gronwall lemma gives the result. □

We now turn to a regularity results for the solution of the drift diffusion system (3.2), which will be needed for the proof of convergence of the next part. We have the

Lemma 3.3. *Under assumptions ($\overline{H1}$)-($\overline{H6}$), there exists a constant C such that the solution of the drift diffusion system* (3.2) *satisfies*

$$\begin{cases}\displaystyle\sum_{0\le p\le 1}\sum_{0\le q\le 3}\sum_{\substack{1\le i,j,k\le 3\\0\le l,m,n\le 3}}\Big\|\nu^q\partial^q_\nu\partial^l_{x_i}\partial^m_{x_j}\partial^n_{x_k}\partial^p_t\rho\Big\|_{L^\infty_t(L^2_{x,\nu,\Omega})}\le C,\\ \displaystyle\sum_{0\le p\le 1}\sum_{\substack{1\le i,j,k\le 3\\0\le l,m,n\le 3}}\Big\|\partial^l_{x_i}\partial^m_{x_j}\partial^n_{x_k}\partial^p_t\bar T\Big\|_{L^\infty_t(L^2_x)}\le C,\end{cases}\tag{3.19}$$

with the convention $\partial^0_X=I_d$, X=x,t, or ν.

Proof. The proof of this lemma is long and tedious, and thus we postpone it to the appendix. It mainly uses the linearity of the equation on ρ and the maximum principle to treat the non linearity of the equation on $\bar T$. □

We now introduce our main result, which is the convergence of the solution of the relativistic transfer equation to the solution of the drift diffusion system as $\varepsilon\to 0$.

Theorem 3.4. *Under assumption* ($\overline{H}1$)-($\overline{H}6$)*, there exists a constant* C*, which does not depend on* ε*, such that the solution of the relativistic transfer system* (3.1) *and the solution of the drift diffusion system* (3.2) *satisfy the following estimate*

$$\|I^\varepsilon(t)-\rho(t)\|_{L^2_{x,\nu,\Omega}}+\|T^\varepsilon(t)-\bar{T}(t)\|_{L^2_x} \leq C\bigg(\|I^\varepsilon(0)-\rho(0)\|_{L^2_{x,\nu,\Omega}}+\|T^\varepsilon(0)-\bar{T}(0)\|_{L^2_x}+\varepsilon\bigg), \ \ 0 \leq t \leq T^f.$$

The end of the paper is devoted to the proof of this convergence result.

3.2.2 Proof of theorem 3.4

In this section the proof of the convergence result (theorem 3.4) is performed. Since it is rather technical, it is divided in three steps. In a first one, the regularity needed on the solution of the drift diffusion system (3.2) to control the remainders of the expansions of the source term (part 3.1.1) is highlighted. In a second part, a function constructed from the truncated Hilbert expansion performed in the part 3.1.2, is proved to be solution of the radiative transfer equations (3.1) with a remainder, which is proved to be small with respect to ε. Finally, in a last part, the difference between this solution and the solution of the radiative transfer equations (3.1) is shown to tends to 0 with ε in L^2. In particular a suitable weight is used to conclude the proof.

First step: control of the remainders of the expansions of the source term

In the last section, the scattering and emission absorption operators (see part 3.1.1) have been expanded in power of ε, but no attention was given to the remainders $\bar{Q}^\varepsilon_{s,i}$, for $i = 0, 1, 2$ defined respectively in section (3.3), (3.4) and (3.5) and to $\bar{Q}^\varepsilon_a$ defined in (3.8). This is the purpose of this part. The following results are important, in the sense that they highlight the regularity needed on the solution of the drift diffusion system to control these remainders. The proof of these results are postponed to the appendix.

Lemma 3.5. *Assume that* J *is a given function. Under assumptions* ($\overline{H}1$)-($\overline{H}4$)*, there exists a constant* $C \geq 0$ *which does not depend on* ε*, such that the following inequality holds*

$$\|\bar{Q}^\varepsilon_{s,i}(J)\|_{L^\infty_t(L^2_{x,\nu,\Omega})} \leq C\sum_{k=0}^{i+1}\|\nu^k\partial^k_\nu J\|_{L^\infty_t(L^2_{x,\nu,\Omega})}, \qquad i = 0, 1, 2.$$

with the convention $\partial^0_\nu = I_d$.

In the same way, the following lemma shows the regularity needed on given functions J and G to control the remainder $\bar{Q}^\varepsilon_a$ of the emission absorption operator.

Lemma 3.6. *Assume that* G *and* J *are two given functions. Under assumptions* ($\overline{H}1$)-($\overline{H}4$)*, there exists a constant* $C \geq 0$ *which depends on the* $L^\infty([0, T^f] \times \mathbb{R}^3_x)$ *norm of* G*, but does not depend on* ε *such that the following inequality holds*

$$\|\bar{Q}^\varepsilon_a(J, G)\|_{L^\infty_t(L^2_{x,\nu,\Omega})} \leq C\bigg(\|G\|_{L^\infty_t(L^2_x)} + \|J\|_{L^\infty_t(L^2_{x,\nu,\Omega})} + \|\nu J\|_{L^\infty_t(L^2_{x,\nu,\Omega})}\bigg).$$

Second step: reconstruction of the solution

In this part is reconstructed a pair $(\hat{I}^\varepsilon, \hat{T})$ constructed from the formal Hilbert expansion, solution to (3.1) with a remainder, which is shown to be small with respect to ε in some norm. These functions are defined by

$$\begin{cases} \hat{I}^\varepsilon = \rho + \varepsilon\rho_1 + \varepsilon^2\rho_2, \\ \hat{T} = \bar{T}, \end{cases} \tag{3.20}$$

where ρ, ρ_1, ρ_2 and $\bar{T}$ are constructed as follow: $(\rho, \bar{T})$ is the solution of the drift-diffusion equation (3.15),

$$\rho_1 = \frac{1}{\sigma_s(x)} \vec{\Omega}.\nabla_x \rho - \vec{\Omega}.\vec{u}\nu\partial_\nu\rho + 3\vec{\Omega}.\vec{u}\rho, \tag{3.21}$$

and

$$\begin{aligned}\rho_2 = &\left((\vec{\Omega},\vec{u})^2 - \frac{|\vec{u}|^2}{3}\right)\left(6\rho - 3\nu\partial_\nu\rho + \nu^2\partial_\nu^2\rho\right) + \frac{1}{\sigma_s^2(x)}\sum_{i,j}(\vec{\Omega}_i\vec{\Omega}_j - <\vec{\Omega}_i\vec{\Omega}_j>)\partial_{x_j}\partial_{x_i}\rho \\ &+ \frac{1}{\sigma_s(x)}\sum_{i,j}(\vec{\Omega}_i\vec{\Omega}_j - <\vec{\Omega}_i\vec{\Omega}_j>)\Big(\big(\nu\partial_\nu\rho - 3\rho\big)\partial_{x_j}\vec{u}_i + \big(\nu\partial_\nu\partial_{x_j}\rho - 3\partial_{x_j}\rho\big)\vec{u}_i\Big) \\ &- \sum_{i,j}(\vec{\Omega}_i\vec{\Omega}_j - <\vec{\Omega}_i\vec{\Omega}_j>)\vec{u}_i\partial_{x_j}\rho.\end{aligned} \tag{3.22}$$

Obviously ρ^1 and ρ^2 are related to the definition of I^1 (3.11) and I^2 (3.14). In this part is proved the following lemma, which shows that $(\hat{I}^\varepsilon, \hat{T})$ is solution of the radiative transfer equations (3.1) with remainders R^ε and S^ε.

Lemma 3.7. *Under assumptions $(\overline{H1})$-$(\overline{H6})$, the pair $(\hat{I}^\varepsilon, \hat{T})$ previously constructed is solution of the following system*

$$\begin{cases} \partial_t \hat{I}^\varepsilon + \frac{1}{\varepsilon}\vec{\Omega}.\nabla_x \hat{I}^\varepsilon = \frac{1}{\varepsilon^2} Q_s^\varepsilon(\hat{I}^\varepsilon) + Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T}) + \varepsilon R^\varepsilon, \\ \partial_t \hat{T} + \nabla.(\hat{T}\vec{u}) + \Gamma\hat{T}\nabla.\vec{u} = -\int_{\nu,\Omega} \frac{\Lambda^\varepsilon}{\gamma^\varepsilon} Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T}) + \varepsilon S^\varepsilon, \end{cases} \tag{3.23}$$

where R_ε and S_ε are such that there exists a constant C which does not depend on ε such that $\|R^\varepsilon\|_{L_t^\infty(L^2_{x,\nu,\Omega})} \le C$ and $\|S^\varepsilon\|_{L_t^\infty(L^2_x)} \le C$.

Proof. Using the results of the previous section, it is easy to see that

$$\begin{cases} Q_s^0(\rho) = 0, \\ \vec{\Omega}.\nabla\rho = Q_s^0(\rho_1) + Q_s^1(\rho), \\ \partial_t\rho + \vec{\Omega}.\nabla\rho_1 = Q_s^0(\rho_2) + Q_s^1(\rho_1) + Q_s^2(\rho) + Q_a^0(\rho, \hat{T}). \end{cases}$$

We write $Q_s^\varepsilon(\hat{I}^\varepsilon)$ as

$$Q_s^\varepsilon(\hat{I}^\varepsilon) = Q_s^0(\rho) + \varepsilon\big(Q_s^0(\rho_1) + Q_s^1(\rho)\big) + \varepsilon^2\big(Q_s^0(\rho_2) + Q_s^1(\rho_1) + Q_s^2(\rho)\big) + \varepsilon^3\big(\bar{Q}^\varepsilon_{s,2}(\rho) + \bar{Q}^\varepsilon_{s,1}(\rho_1) + \bar{Q}^\varepsilon_{s,0}(\rho_2)\big)$$

Using algebraic arguments, we obtain the system (3.23), where

$$R^\varepsilon = \partial_t\rho_1 + \varepsilon\partial_t\rho_2 + \vec{\Omega}.\nabla\rho_2 - \big(\bar{Q}^\varepsilon_{s,2}(\rho) + \bar{Q}^\varepsilon_{s,1}(\rho_1) + \bar{Q}^\varepsilon_{s,0}(\rho_2)\big) + \frac{Q_a^0(\rho,\hat{T}) - Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T})}{\varepsilon}, \tag{3.24}$$

and $S^\varepsilon = \frac{1}{\varepsilon}\int_{\nu,\Omega}\big(\frac{\Lambda}{\gamma}Q_a^\varepsilon(\hat{I}^\varepsilon,\hat{T}) - Q_a^0(\rho,\hat{T})\big)$. We now study the remainders R^ε and S^ε. First, using the lemma 3.5, one has

$$\begin{aligned}\left\|\bar{Q}^\varepsilon_{s,2}(\rho) + \bar{Q}^\varepsilon_{s,1}(\rho_1) + \bar{Q}^\varepsilon_{s,0}(\rho_2)\right\|_{L_t^\infty(L^2_{x,\nu,\Omega})} \le C\Bigg(&\sum_{k=0}^3 \|\nu^k\partial_\nu^k\rho\|_{L_t^\infty(L^2_{x,\nu,\Omega})} + \sum_{k=0}^2 \|\nu^k\partial_\nu^k\rho_1\|_{L_t^\infty(L^2_{x,\nu,\Omega})} \\ &+ \sum_{k=0}^1 \|\nu^k\partial_\nu^k\rho_2\|_{L_t^\infty(L^2_{x,\nu,\Omega})}\Bigg).\end{aligned}$$

The definitions of ρ_1 (3.21) and ρ_2 (3.22) yield, with another constant C,

$$\|R^\varepsilon\|_{L^\infty_t(L^2_{x,\nu,\Omega})} \leq C\Big(\sum_{0\leq p\leq 1} \sum_{0\leq q\leq 3} \sum_{\substack{1\leq i,j,k\leq 3\\ 0\leq l,m,n\leq 3}} \Big\| \nu^q \partial^q_\nu \partial^l_{x_i} \partial^m_{x_j} \partial^n_{x_k} \partial^p_t \rho \Big\|_{L^\infty_t(L^2_{x,\nu,\Omega})} + \Big\| \frac{Q^0_a(\rho,\hat{T}) - Q^\varepsilon_a(\hat{I}_\varepsilon,\hat{T})}{\varepsilon} \Big\|_{L^\infty_t(L^2_{x,\nu,\Omega})} \Big).$$

The first term is uniformly bounded thanks to the lemma 3.19. For the second one, one has

$$\frac{Q^0_a(\rho,\hat{T}) - Q^\varepsilon_a(\hat{I}^\varepsilon,\hat{T})}{\varepsilon} = -\bar{Q}^\varepsilon_a(\hat{I}^\varepsilon,\hat{T}) + \sigma_a(\nu)\big(\rho_1 + \varepsilon\rho_2\big),$$

where $\bar{Q}^\varepsilon_a$ is defined in (3.8). Since a maximum principle has been provided, the lemma 3.6 can be applied to control the first term of the right member. Using the definition of ρ_1 (3.21), ρ_2 (3.22) and the lemma 3.3 to control the other ones, one easily gets

$$\Big\| \frac{Q^0_a(\rho,\hat{T}) - Q^\varepsilon_a(\hat{I}^\varepsilon,\hat{T})}{\varepsilon} \Big\|_{L^\infty_t(L^2_{x,\nu,\Omega})} \leq C,$$

where the constant C is uniform in ε. It yields, with another (uniform in ε) constant C,

$$\|R^\varepsilon\|_{L^\infty_t(L^2_{x,\nu,\Omega})} \leq C. \tag{3.25}$$

We now look at the term S_ε. One can rewrite it as

$$\begin{aligned} S^\varepsilon = & \frac{1}{\varepsilon}\int_{\nu,\Omega} \Big(\frac{\Lambda}{\gamma} - 1\Big)\Big(\sigma_a(\nu_0)\Big(B(\nu_0,\hat{T}) - \hat{I}^\varepsilon\Big)\Big) && := S^\varepsilon_1 \\ & - \frac{1}{\varepsilon}\int_{\nu,\Omega} \Big(\frac{\Lambda}{\gamma} - 1\Big)\Big(\sigma_a(\nu)\Big(B(\nu,\hat{T}) - \rho\Big)\Big) && := S^\varepsilon_2 \\ & + \frac{1}{\varepsilon}\int_{\nu,\Omega} \Big(\sigma_a(\nu_0)B(\nu_0,\hat{T}) - \sigma_a(\nu)B(\nu,\hat{T})\Big) && := S^\varepsilon_3 \\ & + \frac{1}{\varepsilon}\int_{\nu,\Omega} \Big(\sigma_a(\nu)\rho - \sigma_a(\nu_0)\hat{I}^\varepsilon\Big) && := S^\varepsilon_4 \end{aligned}$$

We study each term separately. By definition, $\Lambda\gamma^{-1} - 1 = \varepsilon(\vec{\Omega},\vec{u})$. The relation $(a+b)^2 \leq 2(a^2+b^2)$ yields

$$\|S^\varepsilon_1\|^2_{L^2_x} \leq 2\|\vec{u}\|^2_{L^\infty_x}\Big\{ \int_x \Big(\int_\nu \sigma_a(\nu_0)B(\nu_0,\hat{T})\Big)^2 + \int_x \Big(\int_{\nu,\Omega} \sigma_a(\nu_0)\hat{I}^\varepsilon\Big)^2\Big\}.$$

A Cauchy Schwarz inequality in the second term gives

$$\|S^\varepsilon_1\|^2_{L^2_x} \leq 2\|\vec{u}\|^2_{L^\infty_x}\Big\{ \|\sigma_a\|^2_{L^\infty_x} \int_x \Big(\int_\nu B(\nu_0,\hat{T})\Big)^2 + \|\sigma_a\|^2_{L^2_\nu}\|\hat{I}^\varepsilon\|^2_{L^2_{x,\nu,\Omega}}\Big\},$$

which is bounded thanks to the assumption $\overline{\text{H4}}$ on the emission absorption coefficient. The definition of the Planck function, together with the change of variable $\nu \mapsto \nu_0/\hat{T}$, yield $\int_\nu B(\nu_0,\hat{T}) = \hat{T}^4\int_\nu \nu^3\big(e^\nu - 1\big)^{-1} \leq C\hat{T}^4$. It yields in particular $\int_x \big(\int_\nu B(\nu_0,\hat{T})\big)^2 \leq C\|\hat{T}\|_{L^2_x}$. Finally, the maximum principle (lemma 3.2), the definition of $\hat{I}$ in terms of ρ, ρ_1 and ρ_2 (3.20) and the regularity of the solution of the drift diffusion system (lemma 3.19) yield $\|S^\varepsilon_1\|_{L^2_x} \leq C$. The same arguments give $\|S^\varepsilon_2\|_{L^2_x} \leq C$. The control of the term S^ε_3 is a little more technical. In order to use the fact that $\nu_0 - \nu$ is of order ε, we decompose it as

$$\begin{aligned} S^\varepsilon_3 &= \frac{1}{\varepsilon}\int_{\nu,\Omega}\Big(\Big(\sigma_a(\nu_0) - \sigma_a(\nu)\Big)B(\nu_0,\hat{T}) + \Big(B(\nu_0,\hat{T}) - B(\nu,\hat{T})\Big)\sigma_a(\nu)\Big) \\ &= S^\varepsilon_{3,1} + S^\varepsilon_{3,2}. \end{aligned}$$

The relations $\sigma_a(\nu_0) - \sigma_a(\nu) = \int_\nu^{\nu_0} \sigma_a'(s)ds$ and $\nu - \nu_0 = (1-\Lambda)\nu$ yield

$$\|S_{3,1}^\varepsilon\|_{L_x^2}^2 \le \|\sigma_a\|_{W^{1,\infty}}^2 \frac{\sup\limits_{t,x} |1-\Lambda|^2}{\varepsilon^2} \int_x \Big(\int_\nu \nu B(\nu_0, \hat{T}) \Big)^2.$$

The definition of the Planck function, together with the change of variable $\nu \mapsto \nu_0/\hat{T}$ give us $\int_\nu \nu B(\nu_0, \hat{T}) = \Lambda^{-2}\hat{T}^5 \int_\nu \nu^4 \big(e^\nu - 1\big)^{-1} \le C\hat{T}^5$. Finally, the relation $|1-\Lambda| \le C\varepsilon$, the assumption $\overline{H4}$ on the regularity of the emission absorption coefficient and the maximum principle (lemma 3.2) lead to $\|S_{3,1}^\varepsilon\|_{L_x^2}^2 \le C$. For the term $S_{3,2}^\varepsilon$ the relation $B(\nu_0, \hat{T}) - B(\nu, \hat{T}) = \int_\nu^{\nu_0} \partial_\nu B(s, \hat{T})ds$ gives

$$\|S_{3,2}^\varepsilon\|_{L_x^2}^2 = \frac{1}{\varepsilon^2} \int_x \Big(\int_{\nu,\Omega} \int_{s=\nu}^{s=\nu_0} \sigma_a(\nu) \partial_\nu B(s, \hat{T}) \Big)^2.$$

The Fubini's theorem yields $\int_\nu \int_{s=\nu}^{s=\Lambda\nu} \sigma_a(\nu)\partial_\nu B(s,\hat{T}) = \int_s \int_{\nu=s/\Lambda}^{\nu=s} \sigma_a(\nu)\partial_\nu B(s,\hat{T})$. One thus finds, using the estimate (3.16)

$$\|S_{3,2}^\varepsilon\|_{L_x^2}^2 \le \frac{\sup\limits_{t,x} |1-\Lambda|^2}{\varepsilon^2} \frac{\|\sigma_a\|_{L_\nu^\infty}}{\Lambda_*} \int_x \Big(\int_\nu \nu \partial_\nu B(\nu, \hat{T}) \Big)^2.$$

The definition of the Planck function gives $\nu\partial_\nu B(\nu,\hat{T}) = (3 - (1-e^{-\nu/\hat{T}})^{-1}\nu/\hat{T})B(\nu,\hat{T})$. Once again, the change of variable $\nu \mapsto \nu_0/\hat{T}$ give us $\int_\nu \nu\partial_\nu B(\nu,\hat{T}) \le C$, which yields, with another constant C, $\|S_{3,2}^\varepsilon\|_{L_x^2} \le C$. The same arguments give $\|S_4^\varepsilon\|_{L_x^2} \le C$, which concludes the proof. $\square$

Third step: end of the proof

In this part we end the proof of the theorem 3.4. The following lemma shows that for all $t \in [0, T^f]$, the function $\|I^\varepsilon(t) - \hat{I}^\varepsilon(t)\|_{L^2_{x,\nu,\Omega}} + \|T^\varepsilon(t) - \hat{T}(t)\|_{L^2_{x,}}$ tends to 0 with ε. The important point of the proof is the use of a well chosen weight to overcome the fact that the scattering operator is of order ε^{-2}.

Lemma 3.8. *Under assumptions ($\overline{H1}$)-($\overline{H6}$), there exist a constant C which does not depend on ε such that $(I^\varepsilon - \hat{I}^\varepsilon, T^\varepsilon - \hat{T})$ satisfy the following estimate*

$$\|I^\varepsilon(t) - \hat{I}^\varepsilon(t)\|_{L^2_{x,\nu,\Omega}} + \|T^\varepsilon(t) - \hat{T}(t)\|_{L_x^2} \le C\Big(\|I^\varepsilon(0) - \hat{I}^\varepsilon(0)\|_{L^2_{x,\nu,\Omega}} + \|T^\varepsilon(0) - \hat{T}(0)\|_{L_x^2} + \varepsilon \Big), 0 \le t \le T^f.$$

Proof. We denote $E^\varepsilon = I^\varepsilon - \hat{I}^\varepsilon$ and $F^\varepsilon = T^\varepsilon - \hat{T}$. Using the lemma 3.7, the couple $(E^\varepsilon, F^\varepsilon)$ satisfies the following system :

$$\begin{cases} \partial_t E^\varepsilon + \frac{1}{\varepsilon}\vec{\Omega}.\nabla_x E^\varepsilon = \frac{1}{\varepsilon^2} Q_s^\varepsilon(E^\varepsilon) - \varepsilon R^\varepsilon + U^\varepsilon, \\ \partial_t F^\varepsilon + \nabla.(F^\varepsilon \vec{u}) + \Gamma F^\varepsilon \nabla.\vec{u} = -\varepsilon S^\varepsilon + V^\varepsilon, \end{cases} \tag{3.26}$$

with

$$\begin{cases} U^\varepsilon = Q_a^\varepsilon(I^\varepsilon, T^\varepsilon) - Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T}), \\ V^\varepsilon = -\int_{\nu,\Omega} \frac{\Lambda}{\gamma}\Big(Q_a^\varepsilon(I^\varepsilon, T^\varepsilon) - Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T}) \Big). \end{cases}$$

Let us first prove that the $L^2_{x,\nu}$ norm of U^ε and the L_x^2 norm of V^ε are controlled by the $L^2_{x,\nu}$ norm of E^ε and the L_x^2 norm of F^ε. The definition of the emission absorption operator (3.6) yields

$$U^\varepsilon = \sigma_a(\nu_0)\Big(\big(B(\nu_0, T^\varepsilon) - B(\nu_0, \hat{T})\big) - \Lambda^3\big(I^\varepsilon - \hat{I}^\varepsilon\big) \Big)$$

The triangular inequality together with the estimate (3.16) yield

$$\|U^\varepsilon\|_{L^2_{x,\nu,\Omega}} \leq \ \max\big(1,\Lambda^{*3}\big)\|\sigma_a\|_{L^\infty_x}\bigg(\|B(\nu,T^\varepsilon)-B(\nu,\hat{T})\|_{L^2_{x,\nu}}+\|I^\varepsilon-\hat{I}^\varepsilon\|_{L^2_{x,\nu,\Omega}}\bigg).$$

The same arguments than in the proof of the lemma 2.4 give $\|B(\nu,T^\varepsilon)-B(\nu,\hat{T})\|_{L^2_{x,\nu}} \leq C\|T^\varepsilon - \hat{T}\|_{L^2_x}$, which yields

$$\|U^\varepsilon\|_{L^2_{x,\nu,\Omega}} \leq C\bigg(\|E^\varepsilon\|_{L^2_{x,\nu,\Omega}}+\|F^\varepsilon\|_{L^2_x}\bigg). \tag{3.27}$$

We turn to the control of V^ε. By definition of the operator Q_a^ε (1.8), one has

$$V^\varepsilon = \int_{\nu,\Omega}\frac{\Lambda^2}{\gamma}\sigma_a(\nu_0)\bigg(\frac{B(\nu_0,\hat{T})-B(\nu_0,T^\varepsilon)}{\Lambda^3}+I^\varepsilon-\hat{I}^\varepsilon\bigg).$$

We treat these terms separately. For the first one, the estimate 3.16, the assumption $\overline{\text{H4}}$ on the regularity of the emission absorption coefficient and the arguments used to control U^ε (3.27) give

$$\bigg\|\int_{\nu,\Omega}\frac{\Lambda^2}{\gamma}\sigma_a(\nu_0)\frac{B(\nu_0,\hat{T})-B(\nu_0,T^\varepsilon)}{\Lambda^3}\bigg\|_{L^2_x} \leq (\Lambda^*)^{-1}C\|\hat{T}-T^\varepsilon\|_{L^2_x},$$

For the second one, a Cauchy Schwarz inequality yield

$$\bigg\|\int_{\nu,\Omega}\frac{\Lambda^2}{\gamma}\sigma_a(\nu_0)\big(I^\varepsilon-\hat{I}^\varepsilon\big)\bigg\|_{L^2_x} \leq C\|\sigma_a\|_{L^2_\nu}\|I^\varepsilon-\hat{I}^\varepsilon\|_{L^2_x}.$$

It finally yields, with another constant C

$$\|V^\varepsilon\|_{L^2_x} \leq C\bigg(\|T^\varepsilon-\hat{T}\|_{L^2_x}+\|I^\varepsilon-\hat{I}^\varepsilon\|_{L^2_{x,\nu,\Omega}}\bigg). \tag{3.28}$$

To obtain the proposed result, we need a stability result for the modified transfer system (3.26). The problematic term comes from the scattering operator, which is of order ε^{-2}. Since the scattering operator is isotropic in the moving frame, we multiply the first equation of (3.26) by the weight $\Lambda\gamma^{-1}E_0^\varepsilon$, with an obvious notation $E_0^\varepsilon = \Lambda^3 E^\varepsilon$ and we integrate it on $\mathbb{R}^3_x\times\mathbb{R}^+_\nu\times S^2$. This gives us

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{x,\nu,\Omega}\gamma^{-1}\Lambda^4(E^\varepsilon)^2 &= \int_{x,\nu,\Omega}\gamma^{-1}\Lambda^4E^\varepsilon\big(-\varepsilon R^\varepsilon+U^\varepsilon\big)+\int_x\gamma^{-1}\sigma_s(x)\int_{\nu,\Omega}\frac{1}{\Lambda}E_0^\varepsilon\big(\langle E_0^\varepsilon\rangle_0-E_0^\varepsilon\big)\\ &+\frac{1}{2}\int_{x,\nu,\Omega}(E^\varepsilon)^2\bigg(\partial_t\gamma^{-1}\Lambda^4+\frac{1}{\varepsilon}\vec{\Omega}.\nabla\gamma^{-1}\Lambda^4\bigg),\end{aligned} \tag{3.29}$$

where we reminds that $< E_0^\varepsilon >= \int_{\Omega_0}E_0^\varepsilon$. Using the invariance of the measure $\nu d\nu d\vec{\Omega} = \nu_0 d\nu_0 d\vec{\Omega}_0$, one has

$$\int_{\nu,\Omega}\frac{1}{\Lambda}E_0^\varepsilon\big(\langle E_0^\varepsilon\rangle_0-E_0^\varepsilon\big) = \int_{\nu_0}\bigg(\langle E_0^\varepsilon\rangle_0^2-\langle E_0^{\varepsilon 2}\rangle_0\bigg),$$

which is non positive thanks to a Cauchy-Schwarz inequality, and it shows the importance of the chosen weight. Adding equation (3.29) with the second equation of (3.26) multiplied by F^ε and integrating on $\mathbb{R}^3_x$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\bigg(\int_{x,\nu,\Omega}\gamma^{-1}\Lambda^4(E^\varepsilon)^2+\int_x(F^\varepsilon)^2\bigg) &\leq \frac{1}{2}\int_{x,\nu,\Omega}(E^\varepsilon)^2\bigg|\partial_t\gamma^{-1}\Lambda^4+\frac{1}{\varepsilon}\vec{\Omega}.\nabla\gamma^{-1}\Lambda^4\bigg|+\int_x(F^\varepsilon)^2|\nabla.\vec{u}|\bigg(\frac{1}{2}+\Gamma\bigg)\\ &+\int_{x,\nu,\Omega}\gamma^{-1}\Lambda^4|E^\varepsilon|\big|U^\varepsilon+\varepsilon R^\varepsilon\big|+\int_x|F^\varepsilon|\big|V^\varepsilon+\varepsilon S^\varepsilon\big|.\end{aligned}$$

We need to control the two last terms. Using the inequalities $ab \leq 1/2(a^2+b^2)$ and $(a+b)^2 \leq 2(a^2+b^2)$, one has

$$\int_{x,\nu,\Omega} \gamma^{-1}\Lambda^4 |E^\varepsilon| \big| U^\varepsilon + \varepsilon R^\varepsilon \big| \leq \Lambda^{*4}\bigg(\frac{1}{2}\|E^\varepsilon\|^2_{L^2_{x,\nu,\Omega}} + \|U^\varepsilon\|^2_{L^2_{x,\nu,\Omega}} + \varepsilon^2 \|R^\varepsilon\|^2_{L^2_{x,\nu,\Omega}} \bigg).$$

Using the estimate (3.27), one finally finds

$$\int_{x,\nu,\Omega} \gamma^{-1}\Lambda^4 |E^\varepsilon| \big| U^\varepsilon + \varepsilon R^\varepsilon \big| \leq C\bigg(\|E^\varepsilon\|^2_{L^2_{x,\nu,\Omega}} + \|F^\varepsilon\|^2_{L^2_{x,\nu}} + \varepsilon^2 \|R^\varepsilon\|^2_{L^2_{x,\nu,\Omega}} \bigg),$$

where the constant C is uniform in ε. One also has

$$\int_x |F^\varepsilon| \big| V^\varepsilon + \varepsilon S^\varepsilon \big| \leq \frac{1}{2}\|F^\varepsilon\|^2_{L^2_x} + \|V^\varepsilon\|^2_{L^2_x} + \varepsilon^2 \|S^\varepsilon\|^2_{L^2_x},$$

which gives us, together with estimate (3.28)

$$\int_x |F^\varepsilon| \big| V^\varepsilon + \varepsilon S^\varepsilon \big| \leq C\bigg(\|F^\varepsilon\|^2_{L^2_x} + \|E^\varepsilon\|^2_{L^2_{x,\nu}} + \varepsilon^2 \bigg).$$

This gives us

$$\frac{1}{2}\frac{d}{dt}\bigg(\int_{x,\nu,\Omega} \gamma^{-1}\Lambda^4 (E^\varepsilon)^2 + \int_x (F^\varepsilon)^2 \bigg) \leq C\bigg(\int_{x,\nu,\Omega} (E^\varepsilon)^2 + \int_x (F^\varepsilon)^2 + \varepsilon^2 \bigg),$$

where the constant C is uniform in ε. Integrating this inequality between 0 and t and using the Gronwall lemma, one gets a new constant C such that

$$\|E^\varepsilon(t)\|_{L^2_{x,\nu,\Omega}} + \|F^\varepsilon(t)\|_{L^2_x} \leq C\bigg(\|E^\varepsilon(0)\|_{L^2_{x,\nu,\Omega}} + \|F^\varepsilon(0)\|_{L^2_x} + \varepsilon \bigg),$$

where the constant C depends on $\min\big(1\,,\ \inf_{t,x,\nu,\Omega} \gamma^{-1}\Lambda^4\big)^{-1}$. Using the estimate (3.16), and the assumption $\overline{\text{H2}}$, which yields the positivity of the Lorentz factor γ, one sees that $\inf_{t,x,\nu,\Omega} \gamma^{-1}\Lambda^4 > 0$, and this concludes the proof. □

Acknowledgements The author warmly thank Bruno Després and Christophe Buet for valuable comments and discussions.

A Appendix: Scattering and emission absorption operator's expansion

In this part we prove the lemma 3.5, which deals with the control of the remainders $\bar{Q}^\varepsilon_{s,i}$, $i = 0,1,2$ of the expansion of the scattering operator at order i defined respectively in section (3.5), (3.4) and (3.3), and the lemma 3.6, which deals with the control of the remainder $\bar{Q}^\varepsilon_a$ of the expansion of the emission absorption operator defined in (3.8). These proofs need the following lemmas. The first one (lemma A.1) provides integrability results for Planck type function, while the second (lemma A.2) concerns the regularity of the coefficients λ_i (study of Q^ε_s in section 3.1.1) and the remainder of their expansions.

Lemma A.1. *For any given function $T \in L^\infty(\mathbb{R}^3_x) \cap L^2(\mathbb{R}^3_x)$ and for all $\alpha \in \mathbb{N}$, there exists a constant $C \geq 0$ such that the following estimate holds $\|\nu^\alpha B(\nu,T)\|_{L^2_{x,\nu}} \leq C\|T\|^{\alpha+\frac{5}{2}}_{L^\infty_x}\|T\|_{L^2_x}$.*

Proof. We remind that $B(\nu,T)=\nu^3\big(e^{\nu/T}-1\big)^{-1}$. The change of variable $y\mapsto \nu/T$ leads to

$$\|\nu^\alpha B(\nu,T)\|^2_{L^2_{x,\nu}}=\int_{x,\nu}\frac{\nu^{6+2\alpha}}{\big(e^{\nu/T}-1\big)^2}d\nu dx=\int_x T^{7+2\alpha}\int_y\frac{y^{6+2\alpha}}{\big(e^y-1\big)^2}dydx.$$

If we denote $f(y)=y^{6+2\alpha}\big(e^y-1\big)^{-2}$, one has $f(0)=0$ and f has an exponential decay as $y\to+\infty$. Thus, there exists a bounded constant C such that $\|f\|_{L^1_\nu}=C$. It yields the existence of a constant C such that $\|\nu^\alpha B(\nu,T)\|_{L^2_{x,\nu}}\le C\|T\|^{\alpha+\frac{5}{2}}_{L^\infty_x}\|T\|_{L^2_x}$, which is the announced result. □

We prove the following lemma, which shows that under assumptions on the coefficients $\vec{u},\sigma_s,\sigma_a$ and ε, the coefficients λ_i (study of Q^ε_s in section 3.1.1) are uniformly bounded in L^∞.

Lemma A.2. *Under assumptions ($\overline{H1}$)-($\overline{H4}$), there exists a constant $C\ge 0$, which does not depend on ε, such that the following estimates holds*

$$\forall i\in[1,6],\ \ \|\lambda_i\|_{L^\infty_{t,x,\Omega,\Omega'}}\le C,\ \ \|R_{\frac{\Lambda}{\Lambda'},2}\|_{L^\infty_{t,x,\Omega,\Omega'}}\le C,\ \ \|R_{\frac{\Lambda'}{\Lambda^3},2}\|_{L^\infty_{t,x,\Omega,\Omega'}}\le C\ ,\ \|R_{\Lambda,2}\|_{L^\infty_{t,x,\Omega,\Omega'}}\le C.$$

Proof. Using the expression of the λ_i, $i\in[1,6]$, the first point is obvious. One has $R_{\frac{\Lambda}{\Lambda'},2}=\frac{(\vec{\Omega}',\vec{u})^2(\vec{\Omega}'-\vec{\Omega},\vec{u})}{1-\varepsilon\vec{\Omega}'.\vec{u}}$, and thus $R_{\frac{\Lambda}{\Lambda'},2}\in L^\infty_{t,x,\Omega,\Omega'}$ using assumptions $\overline{H}$ on $\vec{u}$ and ε. Furthermore, on has

$$\begin{aligned}R_{\frac{\Lambda'}{\Lambda^3},2}&=\frac{1-|\vec{u}|^2}{(1-\varepsilon\vec{\Omega}.\vec{u})^3}\Big(3\vec{\Omega}'.\vec{u}-3\varepsilon(\vec{\Omega},\vec{u})^2+2\varepsilon^2(\vec{\Omega},\vec{u})^3-3(\vec{\Omega},\vec{u})^2\big(3\vec{\Omega}.\vec{u}-3\varepsilon(\vec{\Omega},\vec{u})^2+\varepsilon^2(\vec{\Omega},\vec{u})^3\big)\Big)\\&\quad-|\vec{u}|^2\Big(\big(3\vec{\Omega}-\vec{\Omega}',\vec{u}\big)+3\varepsilon\vec{\Omega}.\vec{u}\big(2\vec{\Omega}-\vec{\Omega}',\vec{u}\big)\Big)\end{aligned}$$

And thus $R_{\frac{\Lambda'}{\Lambda^3},2}\in L^\infty_{t,x,\Omega,\Omega'}$ using assumptions $\overline{H}$ on $\vec{u}$ and ε. We remind that

$$R_{\Lambda,2}=\frac{1}{\varepsilon^3\sqrt{1-\varepsilon^2|\vec{u}|^2}}\int_{1-\varepsilon^2|\vec{u}|^2}^1\frac{1-\varepsilon^2|\vec{u}|^2-s}{4}\frac{1}{s\sqrt{s}}ds+\frac{|\vec{u}|^2}{2\sqrt{1-\varepsilon^2|\vec{u}|^2}}\Big(\varepsilon\frac{|\vec{u}|^2}{2}-\vec{\Omega}.\vec{u}\Big)$$

We have $|R_{\Lambda,2}|\le\frac{\varepsilon|\vec{u}|^4}{4(1-\varepsilon^2|\vec{u}|^2)^2}+\frac{|\vec{u}|^2}{2\sqrt{1-\varepsilon^2|\vec{u}|^2}}\Big(\varepsilon\frac{|\vec{u}|^2}{2}-\vec{\Omega}.\vec{u}\Big)$, and thus one can see that it is bounded in $L^\infty_{t,x,\Omega,\Omega'}$ uniformly with respect to ε. □

We now prove the lemma 3.5. Since the arguments are similar, the proof is provided only for the remainder $\bar{Q}_{s,2}$ of the expansion of the scattering operator at the order 2.

Proof of lemma 3.5. Studying the expression of $\bar{Q}^\varepsilon_{s,2}(I)$ (3.3), one can see that the only complicated terms come from $R_{I,2}$ and $I(\nu',\vec{\Omega}')-I(\nu,\vec{\Omega}')$. For the first one, one has

$$R_{I,2}=\nu R_{\frac{\Lambda}{\Lambda'},2}\partial_\nu I(\nu,\vec{\Omega}')+\frac{\nu^2}{2\varepsilon^3}\Big((\frac{\Lambda}{\Lambda'}-1)^2-(\varepsilon\lambda_5)^2\Big)\partial^2_\nu I(\nu,\vec{\Omega}')+\frac{1}{\varepsilon^3}\int_\nu^{\nu'}\frac{(\nu'-s)^2}{2}\partial^3_\nu I(s,\vec{\Omega}')ds$$

First, one can check that $(\frac{\Lambda}{\Lambda'}-1)^2-(\varepsilon\lambda_5)^2\le C\varepsilon^3$. We have to estimate $\int_{\Omega'}\int_\nu^{\nu'}\frac{(\nu'-s)^2}{2}\partial^3_\nu I_\varepsilon(s,\vec{\Omega}')ds$ in $L^2_{x,\nu,\Omega}$. Using a Hölder inequality, the definition $\nu'=(\Lambda/\Lambda')\nu$ and the estimate $\Lambda/\Lambda'-1\le C\varepsilon$,

$$\int_{x,\nu,\Omega}\Big|\int_{\Omega'}\int_\nu^{\nu'}\frac{(\nu'-s)^2}{2}\partial^3_\nu I(s,\vec{\Omega}')ds\Big|^2\le\frac{\varepsilon^5}{4}\int_{x,\Omega,\Omega'}\int_\nu\int_\nu^{\nu'}\nu^5\big|\partial^3_\nu I(s,\vec{\Omega}')\big|^2dsd\nu$$

Using Fubini's theorem, we get

$$\int_{x,\nu,\Omega}\Big|\int_{\Omega'}\int_\nu^{\nu'}\frac{(\nu'-s)^2}{2}\partial^3_\nu I(s,\vec{\Omega}')ds\Big|^2\le\frac{\varepsilon^5}{4}\int_{x,\Omega,\Omega'}\int_s\int_{(\Lambda'/\Lambda)s}^s\nu^5\big|\partial^3_\nu I(s,\vec{\Omega}')\big|^2d\nu ds$$

This gives us

$$\int_{x,\nu,\Omega}\left|\int_{\Omega'}\int_{\nu}^{\nu'}\frac{(\nu'-s)^2}{2}\partial_\nu^3 I(s,\vec{\Omega}')ds\right|^2 \leq C\varepsilon^6\|\nu^3\partial_\nu^3 I\|^2_{L^2_{x,\nu,\Omega}}$$

Thus,

$$\left\|\int_{\Omega'}\int_{\nu}^{\nu'}\frac{(\nu'-s)^2}{2}\partial_\nu^3 I(s,\vec{\Omega}')ds\right\|_{L^\infty_t(L^2_{x,\nu,\Omega})} \leq C\varepsilon^3\|\nu^3\partial_\nu^3 I\|_{L^\infty_t(L^2_{x,\nu,\Omega})}$$

Using the same arguments, one can see that

$$\left\|\int_{\Omega'}\frac{I(\nu',\vec{\Omega}')-I(\nu,\vec{\Omega}')}{\varepsilon}\right\|_{L^\infty_t(L^2_{x,\nu,\Omega})} \leq C\|\nu\partial_\nu I\|_{L^\infty_t(L^2_{x,\nu,\Omega})}$$

Using these estimates, the assumption $\overline{H3}$ on the scattering coefficient σ_s and the lemma A.2, one easily finds the result □

We now turn to the proof of lemma 3.6, which deals with the control of the remainder $\bar{Q}_a^\varepsilon$ of the expansion of the emission absorption operator.

Proof of the lemma 3.6. We recall here the remainder $\bar{Q}_a^\varepsilon$

$$\left\{\begin{aligned}\bar{Q}_a^\varepsilon(I,T) = &\frac{1-\Lambda^2}{\varepsilon\Lambda^2}\sigma_a(\nu_0)\big(B(\nu_0,T)-I_0\big) + \frac{\sigma_a(\nu_0)-\sigma_a(\nu)}{\varepsilon}\big(B(\nu_0,T)-I\big)\\ &+\sigma_a(\nu)\frac{B(\nu_0,T)-B(\nu,T)}{\varepsilon} + \sigma_a(\nu_0)\frac{I-I_0}{\varepsilon}.\end{aligned}\right.$$

We start with the first term. One can see that $|1-\Lambda^2| \leq C\varepsilon$, where C is bounded in $L^\infty([0,T^f]\times\mathbb{R}^3_x\times S^2)$ uniformly in ε. One thus has

$$\left\|\frac{1-\Lambda^2}{\varepsilon\Lambda^2}\sigma_a(\nu_0)\big(B(\nu_0,T)-I_0\big)\right\|_{L^\infty_t(L^2_{x,\nu,\Omega})} \leq \frac{C}{\Lambda_*}\|\sigma_a\|_{L^\infty_\nu}\bigg(\|B(\nu_0,T)\|_{L^\infty_t(L^2_{x,\nu,\Omega})} + \|I_0\|_{L^\infty_t(L^2_{x,\nu,\Omega})}\bigg).$$

Using the change of variable $\nu\to\Lambda\nu$ and the relation $I_0 = \Lambda^3 I$, one finds a new constant C such that

$$\left\|\frac{1-\Lambda^2}{\varepsilon\Lambda^2}\sigma_a(\nu_0)\big(B(\nu_0,T)-I_0\big)\right\|^2_{L^\infty_t(L^2_{x,\nu,\Omega})} \leq C\bigg(\|B(\nu,T)\|_{L^\infty_t(L^2_{x,\nu,\Omega})} + \|I\|_{L^\infty_t(L^2_{x,\nu,\Omega})}\bigg).$$

Finally, the lemma A.1 on the integrability of the Planck function (lemma A.1) yield, with another constant C,

$$\left\|\frac{1-\Lambda^2}{\varepsilon\Lambda^2}\sigma_a(\nu_0)\big(B(\nu_0,T)-I_0\big)\right\|_{L^\infty_t(L^2_{x,\nu,\Omega})} \leq C\bigg(\|T\|_{L^\infty_t(L^2_x)} + \|I\|_{L^\infty_t(L^2_{x,\nu,\Omega})}\bigg).$$

We now look at the second component of $\bar{Q}_a^\varepsilon$. One has, using the relation $\sigma_a(\nu_0)-\sigma_a(\nu) = \int_\nu^{\nu_0}\sigma_a'(s)ds$ and a Cauchy-Schwarz inequality,

$$\int_\nu\Big(\big(\sigma_a(\nu_0)-\sigma_a(\nu)\big)\big(B(\nu_0,T)-I\big)\Big)^2 \leq \|\partial_\nu\sigma_a\|^2_{L^\infty_\nu}\int_\nu\big(B(\nu_0,T)-I\big)^2(\nu-\nu_0)^2.$$

By definition $\nu_0-\nu = \nu(\Lambda^\varepsilon-1)$ and thus there exists a constant C such that $|\nu_0-\nu| \leq C\varepsilon\nu$. This estimate, together with a Cauchy-Schwarz inequality and the lemma A.1 give us, with another constant C,

$$\left\|\frac{\sigma_a(\nu_0)-\sigma_a(\nu)}{\varepsilon}\big(B(\nu_0,T)-I\big)\right\|_{L^\infty_t(L^2_{x,\nu,\Omega})} \leq C\|\partial_\nu\sigma_a\|_{L^\infty_\nu}\Big(\|T\|_{L^\infty_t(L^2_x)} + \|\nu I\|_{L^\infty_t(L^2_{x,\nu,\Omega})}\Big)$$

We turn to the third term of $\bar{Q}_a^\varepsilon$. One has, using a Taylor expansion, a Cauchy-Schwarz inequality and the Fubini's theorem,

$$\int_\nu (B(\nu_0,T)-B(\nu,T))^2 d\nu \le \int_\nu \int_\nu^{\nu_0} |\nu_0-\nu||\partial_\nu B(s,T)|^2 ds d\nu \le C\varepsilon \int_s |\partial_\nu B(s,T)|^2 \int_{\frac{s}{\Lambda}}^s \nu d\nu ds$$

As previously, on has $s(1-\frac{1}{\Lambda}) \le C\varepsilon s$ and thus

$$\left\| \sigma_a(\nu) \frac{B(\nu_0,T)-B(\nu,T)}{\varepsilon} \right\|_{L_t^\infty(L^2_{x,\nu,\Omega})} \le C\|\sigma_a\|_{L^\infty_\nu} \|\nu\partial_\nu B\|_{L_t^\infty(L^2_{x,\nu,\Omega})}$$

One has $\nu\partial_\nu B(\nu,T) = 3B(\nu,T) - \frac{\nu}{T}B(\nu,T)\big(1-e^{-\nu/T}\big)^{-1}$. Using the same kind of arguments than for the lemma A.1, one finds $\|\nu\partial_\nu B\|_{L_t^\infty(L^2_{x,\nu,\Omega})} \le C\|T\|_{L_t^\infty(L^2_x)}$. For the last term, the relations $I_0 = \Lambda^3 I$ and $1-\Lambda^3 = C\varepsilon$ yield

$$\left\| \sigma_a(\nu_0) \frac{I-I_0}{\varepsilon} \right\|_{L_t^\infty(L^2_{x,\nu,\Omega})} \le C\|I\|_{L_t^\infty(L^2_{x,\nu,\Omega})},$$

which ends the proof. □

B Regularity of the solution of the drift diffusion system

In this part we prove the lemma 3.3, which deals with the regularity of the solution of the drift diffusion system. The proof mainly uses the linearity of the equation on ρ and the maximum principle to overcome the difficulties coming from the nonlinearity of the equation on T. The proof consists to estimate each terms of the sum in (3.19). Since the proof of the estimates of these terms is rather similar, we only show the development for some of them.

Lemma B.1. *Under assumptions $(\overline{H1})$-$(\overline{H6})$, there exists a constant C such that the solution $(\rho,\bar{T})$ of the drift diffusion system* (3.15) *satisfies the following estimate*

$$\|\rho(t)\|_{L^2_{x,\nu}} + \|\bar{T}(t)\|_{L^2_x} \le C, \qquad 0 \le t \le T^f.$$

Proof. Multiplying the first equation of (3.15) by ρ, integrating on $\mathbb{R}^3_x \times \mathbb{R}^+_\nu$, multiplying the second by T, integrating on $\mathbb{R}^3_x$ and adding, we get

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{x,\nu}\rho^2 + \int_x \bar{T}^2\Big) + \int_{x,\nu}\frac{|\nabla\rho|^2}{6\sigma_s(x)} + \int_{x,\nu}\frac{\rho^2}{2}\nabla.\vec{u} + \big(\Gamma+\frac{1}{2}\big)\int_x \bar{T}^2\nabla.\vec{u}$$
$$= -\int_{x,\nu}\frac{\nabla.\vec{u}}{3}\frac{\rho^2}{2} + \int_{x,\nu}\sigma_a(\nu)\big(B(\nu,\bar{T})-\rho\big)(\rho-\bar{T}).$$

We study the last term. One has

$$\int_{x,\nu}\sigma_a(\nu)\big(B(\nu,\bar{T})-\rho\big)(\rho-\bar{T}) = \int_{x,\nu}\sigma_a(\nu)B(\nu,\bar{T})\rho + \int_{x,\nu}\sigma_a(\nu)\bar{T}\rho - \int_{x,\nu}\sigma_a(\nu)B(\nu,\bar{T})\bar{T} - \int_{x,\nu}\sigma_a(\nu)\rho^2,$$

and we study successively each of those terms. For the first one, the Cauchy-Schwarz inequality yields

$$\int_{x,\nu}\sigma_a(\nu)B(\nu,\bar{T})\rho \le \frac{1}{2}\|\sigma_a\|_{L^\infty_\nu}\big(\|B(\nu,\bar{T})\|^2_{L^2_{x,\nu}} + \|\rho\|^2_{L^2_{x,\nu}}\big).$$

The relation $\|B(\nu,\bar{T})\|_{L^2_{x,\nu}} \le C\|\bar{T}\|_{L^2_x}$ (lemma A.1) and the assumption $\overline{\text{H4}}$ on the regularity of the emission absorption coefficient finally give a constant C such that

$$\int_{x,\nu}\sigma_a(\nu)B(\nu,\bar{T})\rho \le C\big(\|\bar{T}\|^2_{L^2_x} + \|\rho\|^2_{L^2_{x,\nu}}\big).$$

The second one is a litle more complicated. The inequality $ab \leq \frac{1}{2}(a^2+b^2)$ yields

$$\int_{x,\nu} \sigma_a(\nu)\bar{T}\rho \leq \frac{1}{2}\|\bar{T}\|^2_{L^2_x} + \frac{1}{2}\int_x \Big(\int_\nu \sigma_a(\nu)\rho\Big)^2,$$

and a Cauchy Schwarz inequality gives

$$\int_{x,\nu} \sigma_a(\nu)\bar{T}\rho \leq \frac{1}{2}\|\bar{T}\|^2_{L^2_x} + \frac{1}{2}\|\sigma_a\|^2_{L^2_\nu}\|\rho\|^2_{L^2_{x,\nu}}. \tag{B.1}$$

For the third one, one has $\int_{x,\nu}\sigma_a(\nu)B(\nu,\bar{T})\bar{T} \leq \|\sigma_a\|_{L^\infty_\nu}\int_x \bar{T}\int_\nu B(\nu,\bar{T})$. The lemma A.1 and the assumption $\overline{\text{H4}}$ on the regularity of the emission absorption coefficient finally give a constant C such that $\int_{x,\nu}\sigma_a(\nu)B(\nu,\bar{T})\bar{T} \leq C\|\bar{T}\|^2_{L^2_x}$. This gives us, using the assumption $\overline{\text{H3}}$ on the positivity of the scattering coefficient and the assumption $\overline{\text{H1}}$ on the regularity of the velocity field $\vec{u}$ a constant C such that

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{x,\nu}\rho^2 + \int_x \bar{T}^2\Big) \leq C\Big(\int_{x,\nu}\rho^2 + \int_x \bar{T}^2\Big).$$

The Gronwall lemma and the assumption $\overline{\text{H5}}$ on the initial conditions give the expected result. $\square$

The following lemma deals with the control of $\nu\rho$ in $L^2_{x,\nu}$.

Lemma B.2. *Under assumptions ($\overline{H1}$)-($\overline{H6}$), there exists a constant C such that the solution ρ of the first equation of the drift diffusion system* (3.15) *satisfies the following estimate*

$$\|\nu\rho(t)\|_{L^2_{x,\nu}} \leq C, \qquad 0 \leq t \leq T^f.$$

Proof. Multiplying the first equation of (3.15) by $\nu^2\rho$, integrating it on $\mathbb{R}^3_x \times \mathbb{R}^+_\nu$, we get, denoting $h = \nu\rho$,

$$\frac{1}{2}\frac{d}{dt}\int_{x,\nu} h^2 + \int_{x,\nu}\frac{|\nabla h|^2}{6\sigma_s(x)} = -\int_{x,\nu} h^2\frac{\nabla.\vec{u}}{2} + \int_{x,\nu}\sigma_a\big(\nu h B(\nu,\bar{T}) - h^2\big)$$

We look at the last term. One has,

$$\int_{x,\nu}\sigma_a\big(\nu h B(\nu,\bar{T}) - h^2\big) \leq \frac{1}{2}\|\sigma_a\|_{L^\infty_\nu}\Big(\|h\|^2_{L^2_{x,\nu}} + \|\nu B(\nu,\bar{T})\|^2_{L^2_{x,\nu}}\Big).$$

Using the lemma A.1 and the maximum principle (lemma 3.2), one has $\|\nu B(\nu,\bar{T})\|^2_{L^2_{x,\nu}} \leq C\|\bar{T}\|^2_{L^2_x}$. This gives us, with another constant C depending on the L^2_x norm of $\bar{T}$ (lemma B.1), $\frac{d}{dt}\int_{x,\nu}h^2 \leq C(\int_{x,\nu}h^2 + 1)$. The Gronwall lemma and the assumption $\overline{\text{H5}}$ on the initial conditions give the result. $\square$

We now turn to the estimate of $\nu\partial_\nu\rho$. One has the

Lemma B.3. *Under assumptions ($\overline{H1}$)-($\overline{H6}$), there exists a constant C such that the solution ρ of the first equation of the drift diffusion system* (3.15) *satisfies the following estimate*

$$\|\nu\partial_\nu\rho(t)\|_{L^2_{x,\nu}} \leq C, \qquad 0 \leq t \leq T^f.$$

Proof. We differentiate the equation of the drift diffusion system (3.15) with respect to ν, we multiply it by $\nu^2\partial_\nu\rho$ and we integrate it on $\mathbb{R}^3_x \times \mathbb{R}^+_\nu$. Denoting $h = \nu\partial_\nu\rho$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{x,\nu}h^2 + \int_{x,\nu}\frac{|\nabla h|^2}{6\sigma_s(x)} = -\frac{2}{3}\int_{x,\nu}h^2\nabla.\vec{u} + \int_{x,\nu}\partial_\nu\sigma_a\Big(\nu B(\nu,\bar{T}) - \nu\rho\Big)h + \int_{x,\nu}\sigma_a\Big(\nu\partial_\nu B(\nu,\bar{T})h - h^2\Big).$$

Using a Cauchy-Schwarz inequality, the assumptions $\overline{H1}$ on the velocity field, $\overline{H3}$ on the positivity of the scattering coefficient, $\overline{H4}$ on the regularity of the emission absorption coefficient and the lemma B.2 on the control of $\nu\rho$ in $L^2_{x,\nu}$, one gets a constant C such that

$$\frac{1}{2}\frac{d}{dt}\|h(t)\|_{L^2_{x,\nu}} \leq C\bigg(\|h(t)\|^2_{L^2_{x,\nu}} + \|\nu B(\nu,\bar{T})\|_{L^2_{x,\nu}} + \|\nu\partial_\nu B(\nu,\bar{T})\|_{L^2_{x,\nu}}\bigg).$$

Once again, the lemma (A.1) yields $\|\nu B(\nu,\bar{T})\|^2_{L^2_{x,\nu}} \leq C\|\bar{T}\|^2_{L^2_x}$. Furthermore, one has $\nu\partial_\nu B(\nu,\bar{T}) = 3B(\nu,\bar{T}) - \frac{\nu}{\bar{T}}\big(1-e^{-\nu/\bar{T}}\big)^{-1}B(\nu,\bar{T})$. The same arguments than for the proof of the lemma (A.1) yields $\|\nu\partial_\nu B(\nu,\bar{T})\|^2_{L^2_{x,\nu}} \leq C\|\bar{T}\|^2_{L^2_x}$. The Gronwall lemma and the assumption $\overline{\mathrm{H5}}$ on the initial conditions finally give the result. □

Using exactly the same arguments, one can prove under assumptions ($\overline{H1}$)-($\overline{H6}$), that $\forall t \in [0,T^f]$, $\nu^2\partial^2_\nu\rho(t)$ and $\nu^3\partial^3_\nu\rho(t)$ belong to $L^2(\mathbb{R}^3_x \times \mathbb{R}^+_\nu)$. We now turn to the control of the space derivatives of the solution of the diffusion system (3.15). One has the

Lemma B.4. *Under assumptions ($\overline{H1}$)-($\overline{H6}$), there exists a constant C such that the solution $(\rho,\bar{T})$ of the drift diffusion system* (3.15) *satisfies the following estimate: $\|\nabla\rho(t)\|_{L^2_{x,\nu}} + \|\nabla\bar{T}(t)\|_{L^2_x} \leq C, 0 \leq t \leq T^f$.*

proof. Differentiating the first equation of (3.15) with respect to x_j, multiplying the obtained equation by $\partial_{x_j}\rho$ and integrating on $\mathbb{R}^3_x \times \mathbb{R}^+_\nu$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{x,\nu}|\partial_{x_j}\rho|^2 + \int_{x,\nu}\frac{|\nabla\partial_{x_j}\rho|^2}{6\sigma_s(x)} + \frac{1}{3}\int_{x,\nu}\nabla(\partial_{x_j}\rho).\nabla\rho\partial_{x_j}\sigma_s^{-1} + \int_{x,\nu}\partial_{x_j}\rho\nabla.(\rho\partial_{x_j}\vec{u}) = \\ -\frac{2}{3}\int_{x,\nu}|\partial_{x_j}\rho|^2\nabla.\vec{u} + \frac{1}{3}\int_{x,\nu}\partial_{x_j}\rho\nu\partial_\nu\rho\nabla.(\partial_{x_j}\vec{u}) + \int_{x,\nu}\sigma_a\partial_{x_j}\rho\partial_{x_j}\Big(\big(B(\nu,\bar{T})-\rho\big)\Big)\end{aligned}$$

One has $\nabla(\partial_{x_j}\rho).\nabla\rho = \frac{1}{2}\partial_{x_j}|\nabla\rho|^2$. Furthermore,

$$\partial_{x_j}\rho\nabla.(\rho\partial_{x_j}\vec{u}) = \frac{1}{2}\partial_{x_j}\rho^2\nabla.\partial_{x_j}\vec{u} + \partial_{x_j}\rho\sum_i\partial_{x_i}\rho\partial_{x_i}\vec{u}_i$$

This gives us

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{x,\nu}|\partial_{x_j}\rho|^2 + \int_{x,\nu}\frac{|\nabla\partial_{x_j}\rho|^2}{6\sigma_s(x)} + \frac{1}{6}\int_{x,\nu}\partial_{x_j}|\nabla\rho|^2\partial_{x_j}\sigma_s^{-1} + \frac{1}{2}\int_{x,\nu}\partial_{x_j}\rho^2\nabla.\partial_{x_j}\vec{u} + \sum_i\int_{x,\nu}\partial_{x_j}\rho\partial_{x_i}\rho\partial_{x_i}\vec{u}_i \\ = -\frac{2}{3}\int_{x,\nu}|\partial_{x_j}\rho|^2\nabla.\vec{u} + \frac{1}{3}\int_{x,\nu}\partial_{x_j}\rho\nu\partial_\nu\rho\nabla.(\partial_{x_j}\vec{u}) + \int_{x,\nu}\sigma_a\rho\partial_{x_j}\partial_{x_j}\Big(\big(B(\nu,\bar{T})-\rho\big)\Big)\end{aligned}$$

Making the sum of this equation for $j = 1,2,3$, using integration by parts, Cauchy Schwarz inequalities, the assumptions $\overline{H1}, \overline{H3}$ and $\overline{H4}$ on the regularity of the coefficients $\vec{u}, \sigma_s$ and σ_a and the lemmas B.1 and B.2 on the integrability of ρ and $\nu\rho$ in $L^2_{x,\nu}$, we get a constant C such that

$$\frac{1}{2}\frac{d}{dt}\int_{x,\nu}|\nabla\rho|^2 \leq C\Big(1 + \int_{x,\nu}|\nabla\rho|^2\Big) + \sum_j\int_{x,\nu}\sigma_a\partial_{x_j}\rho\partial_{x_j}\Big(\big(B(\nu,\bar{T})-\rho\big)\Big)$$

We need to control the last term. One has

$$\sigma_a\partial_{x_j}\rho\partial_{x_j}\Big(\big(B(\nu,\bar{T})-\rho\big)\Big) = \sigma_a\big(\partial_{x_j}\rho\partial_{x_j}B(\nu,\bar{T}) - |\partial_{x_j}\rho|^2\big)$$

Using the definition of $B(\nu,\bar{T})$, one has $\partial_{x_j}B(\nu,\bar{T}) = \partial_{x_j}\bar{T}\frac{\nu}{\bar{T}^2}\nu^4e^{\nu/\bar{T}}\big(e^{\nu/\bar{T}}-1\big)^{-2}$. Thus, one has

$$\sum_j\int_{x,\nu}\sigma_a\partial_{x_j}\rho\partial_{x_j}B(\nu,\bar{T}) \leq \frac{1}{2}\|\sigma_a\|_{L^\infty_\nu}\bigg(\int_{x,\nu}|\nabla\rho|^2 + \int_x\frac{|\nabla\bar{T}|^2}{\bar{T}^4}\int_\nu\frac{\nu^8e^{2\nu/\bar{T}}}{\big(e^{\nu/\bar{T}}-1\big)^4}\bigg).$$

The change of variable $y = \frac{\nu}{\bar{T}}$ leads to

$$\int_x \frac{|\nabla \bar{T}|^2}{\bar{T}^4} \int_\nu \frac{\nu^8 e^{2\nu/\bar{T}}}{\left(e^{\nu/\bar{T}} - 1\right)^4} = \int_x |\nabla \bar{T}|^2 \bar{T}^5 \int_{\mathbb{R}^+} \frac{y^8 e^{2y}}{\left(e^y - 1\right)^4}.$$

The maximum principle (lemma 3.2) together with the same idea than in the proof of the lemma A.1 shows that

$$\int_x |\nabla \bar{T}|^2 \bar{T}^5 \int_y \frac{y^8 e^{2y}}{\left(e^y - 1\right)^4} \leq C \|\nabla \bar{T}\|^2_{L^2_x}.$$

Finally, one finds another constant C such that

$$\sum_j \int_{x,\nu} \partial_{x_j} \rho \partial_{x_j} \Big(\sigma_a \big(B(\nu, \bar{T}) - \rho \big) \Big) \leq C \Big(\int_x |\nabla \bar{T}|^2 + \int_{x,\nu} |\nabla \rho|^2 \Big).$$

This gives us, with another constant C,

$$\frac{1}{2} \frac{d}{dt} \int_{x,\nu} |\nabla \rho|^2 \leq C \Big(1 + \int_x |\nabla \bar{T}|^2 + \int_{x,\nu} |\nabla \rho|^2 \Big) \tag{B.2}$$

We turn to the equation on $\bar{T}$ in (3.15). Differentiating this equation with respect to x_j, multiplying it by $\partial_{x_j} \bar{T}$, integrating on $\mathbb{R}^3$ and taking the sum for $j = 1, 2, 3$, we get

$$\frac{1}{2} \frac{d}{dt} \int_x |\nabla \bar{T}|^2 + \Big(\frac{1}{2} + \Gamma \Big) \int_x |\nabla \bar{T}|^2 + \sum_j \int_x (\partial_{x_j} T) \bar{T} \nabla.(\partial_{x_j} \vec{u}) = - \sum_j \int_{x,\nu} \sigma_a \partial_{x_j} \bar{T} \partial_{x_j} \Big(\big(B(\nu, \bar{T}) - \rho \big) \Big).$$

One has $\sum_j \int_{x,\nu} \sigma_a \partial_{x_j} \bar{T} \partial_{x_j} B = \sum_j \int_{x,\nu} \sigma_a |\partial_{x_j} \bar{T}|^2 \partial_T B$. Once again, the same arguments than for the lemma A.1 on the integrability of the Planck function show that $\partial_T B(\nu, T) \in L^1_\nu$. One finds a constant C such that $\sum_j \int_{x,\nu} \partial_{x_j} \bar{T} \partial_{x_j} \big(\sigma_a B \big) \leq C \|\sigma_a\|_{L^\infty_\nu} \|\nabla \bar{T}\|^2_{L^2_x}$. Moreover, the Cauchy-Schwarz inequality together with the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ yield

$$\sum_j \int_{x,\nu} \sigma_a \partial_{x_j} \bar{T} \partial_{x_j} \rho \leq \frac{1}{2} \|\sigma_a\|^2_{L^2_\nu} \|\nabla \bar{T}\|^2_{L^2_x} + \frac{1}{4} \|\nabla \rho\|^2_{L^2_{x,\nu}}. \tag{B.3}$$

Finally, an integration by parts yields $\sum_j \int_x (\partial_{x_j} \bar{T}) \bar{T} \nabla.(\partial_{x_j} \vec{u}) \leq \frac{1}{2} \|\vec{u}\|_{W^{2,\infty}_x} \|\nabla \bar{T}\|^2_{L^2_x}$. This gives us another constant C such that

$$\frac{1}{2} \frac{d}{dt} \int_x |\nabla \bar{T}|^2 \leq C \Big(1 + \int_x |\nabla \bar{T}|^2 + \int_{x,\nu} |\nabla \rho|^2 \Big),$$

This result, together with the inequality (B.2), yields another constant C such that

$$\frac{1}{2} \frac{d}{dt} \Big(\|\nabla \bar{T}\|^2_{L^2_x} + \|\nabla \rho\|^2_{L^2_{x,\nu}} \Big) \leq C \Big(1 + \|\nabla \bar{T}\|^2_{L^2_x} + \|\nabla \rho\|^2_{L^2_{x,\nu}} \Big).$$

The Gronwall lemma and the assumption $\overline{\text{H5}}$ on the initial conditions give the result. □

We do not give the proof of the remaining terms in (3.19) since it uses the same arguments.

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